On Hardness of Approximating the Parameterized Clique Problem

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The clique problem

**The Clique problem:**

**Input:** A graph $G=(V,E)$ on $n$ vertices, and a parameter $k$.

**Goal:** Find a $k$-clique in $G$ (or declare "there is no $k$-clique").
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[FGLSS ‘96]: It is NP-hard to find a clique of size $k/2$.
[Håstad ‘99]: For $k=n^{0.99}$ it is NP-hard to find a clique of size $n^{0.01}$. 

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Well, what can I say? Looks like a very hard problem...
Parameterized complexity

**The parameterized k-Clique problem:**

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Now we have the trivial algorithm whose running time is $O(n^k)$.

**Question**: Can we do anything less trivial?

Is there an algorithm whose running time is $f(k) \cdot \text{poly}(n)$?
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Is the $k$-Clique problem fixed-parameter tractable?
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The parameterized $k$-VertexCover problem

For the $k$-VertexCover problem there is an algorithm whose running time is $2^{O(k)} \cdot n^2$. 
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Can we hope for something similar for the *k-Clique* problem?
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Can we hope for something similar for the $k$-Clique problem?

Assuming ETH, **$k$-Clique** cannot be solved in time $f(k) \cdot \text{poly}(n)$. 
Approximating the Clique problem

**Gap-Clique\(k, \, k/2\) problem:**

*Input*: A graph \(G=\langle V, E \rangle\) on \(n\) vertices.

*Goal*: Decide between:

- **YES case**: \(G\) contains a \(k\)-clique.
- **NO case**: \(G\) contains no clique of size \(k/2\)-clique.

**Question**: Can we solve **Gap-Clique** in time \(f(k) \cdot \text{poly}(n)\)?

Is the Gap-Clique problem **fixed-parameter tractable**?
Main Result

In the paper we give evidence that \textit{Gap-Clique}(k, k/2) is not fixed-parameter tractable.

We define a constraint satisfaction problem called \textit{k-DEG-2-SAT}, and show an FPT-reduction

\[
k\text{-DEG-2-SAT} \leq_{\text{FPT}} \text{Gap-Clique}(k, k/2)
\]
Main Result

Definition: $[A \leq_{\text{FPT}} B]$
An FPT-reduction from $A$ to $B$
gets an instance $(x,k)$ of $A$ and outputs an instance $(x',k')$ of $B$
such that

1. $(x,k) \in A$ if and only if $(x',k') \in B$
2. $k'$ depends only on $k$.
3. The running time of the reduction is $f(k) \cdot \text{poly}(n)$. 

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If \( A \leq_{FPT} B \) and \( B \) has a FPT-algorithm, then \( A \) also has an FPT-algorithm.
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\textbf{Caveat:} We do not know the status of the \textit{k-DEG-2-SAT} problem. \textit{Could be fixed-parameter tractable ...}
The k-DEG-2-SAT problem

**The k-DEG-2-SAT problem:**

**Input:** A finite field $\mathbf{F}$ of size $n$, and a system of $k$ quadratic equations over $\mathbf{F}$ in $k$ variables $x_1,...x_k$.

$$p_1(x_1,...x_k)=0, \ldots \quad p_k(x_1,...x_k)=0.$$  

**Goal:** Is there a solution $x_1,...x_k \in \mathbf{F}$ that satisfies all the equations?
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**Fact:** *k-DEG-2-SAT* is NP-complete.
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Some observations:

1. There is a trivial algorithm with running time $O(n^k)$.
2. Using Gröbner bases it is possible to find a solution in the *extension field* of $F$ in FPT-time.
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Note: For each $n$ there are $n^{\text{poly}(k)}$ instances of size $n$. *Doesn’t seem to rule out hardness for FPT-algorithms.*
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**Proof:**
Use algebraic techniques from the proof of the PCP theorem
[AS, ALMSS, FGLSS, LFKN, BLR]
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• Sum-check protocol
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• FGLSS reduction
Open problems

1. Give more evidence that $\text{Gap-Clique}(k, k/2)$ is not fixed-parameter tractable. (Ideally: show $k\text{-Clique} \leq_{\text{FPT}} \text{Gap-Clique}(k, k/2)$)

2. Show $\text{Gap-Clique}(k, k/2) \leq_{\text{FPT}} \text{Gap-Clique}(k, k^{0.9})$.

3. Is $\text{Gap-Clique}(k, \log\log(k))$ fixed-parameter tractable?
Thank You