Lecture 11

Undirected s-t connectivity

linear algebra & random walks

Saving random bits via
random walks

Last time:

random walks on graphs

Cover time:

$C(G) \equiv \max_{v} E[\#\text{ steps to visit all nodes in } G \text{ when start at } v]$

starting points

<u>Thm</u> $\forall G$, $C(G)$ is $O(n \cdot m)$

s-t Connectivity (UST-Conn)

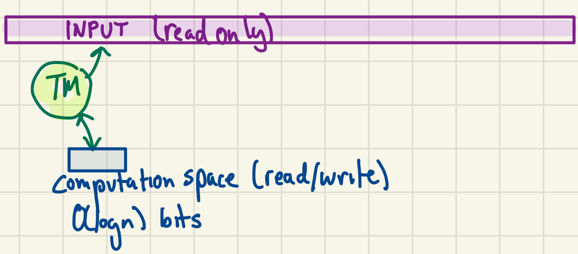
undirected graph

Input undirected G
nodes s,t

Output if s,t in same conn. comp, "Yes"
else "NO"

many ways to solve in poly time
what about space?

RL = class of problems solvable by randomized log-space computations
[no charge for input space (read-only), but can only store const # ptrs]



Thm UST-Conn $\in$ RL

Algorithm:

start at s

take a random walk for $c \cdot n^3$ steps

if ever see t output "Yes"

else output "NO"

Complexity:

time: $O(n^3) \times$ (time to pick random nbr)

space:

Keep track of:

step counter $O(\log n)$

space to pick random nbr:

e.g. scan nbrs & count d_u $O(\log n)$

toss d_u-sided die to get j

scan again to find j^{th} nbr

total $O(\log n)$

Behavior:

If s,t not connected, never output "Yes"

If s,t connected

$$h_{s,t} \le C_s(\underbrace{G_s}_{\text{connected component of } s \text{ (and } t)}) \le n^3$$

$$\Pr[\text{output "no"}] \le \underbrace{\Pr[\text{start at } s, \text{ walk } c \cdot C_s(G_s) \text{ steps \& don't see } t]}$$

$= \Pr[$ time to cover graph is $> c$ times its expectation$]$

$\le \frac{1}{c}$ (Markov's #)

∎

Comments

- Actually USTConn $\in L$!!! (Reingold)

- Open: is STConn for directed graphs $\in L$?

  ($\Rightarrow RL = L$)

  we know $RL \in L^{3/2}$

  recent improvement $RL \in \text{Space}\left(\frac{\log^{3/2} n}{\sqrt{\log\log n}}\right)$

  actually also BPspace(log n)

  due to William Hoza in Random '21

Linear Algebra Review

def v is an eigenvector of A with corresponding eigenvalue λ iff

$$vA = \lambda v$$

def ℓ_2-norm of $v = (v_1 \cdots v_n) = \sqrt{\sum_{i=1}^{n} v_i^2}$

def $v^{(1)} \dots v^{(m)}$ orthonormal if

$$\underbrace{v^{(i)} \cdot v^{(j)}}_{\substack{\text{inner product} \\ = \sum_{\ell} v_\ell^{(i)} \cdot v_\ell^{(j)}}} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

example P = transition matrix of d-reg undir graph (doubly stochastic)

$$\left(\tfrac{1}{n} \;\ldots\; \tfrac{1}{n}\right)\cdot P = 1\cdot\left(\tfrac{1}{n}\cdots\tfrac{1}{n}\right)$$

doesn't this seem more natural? its the probability distribution vector

also $$\left(\tfrac{1}{\sqrt{n}} \;\ldots\; \tfrac{1}{\sqrt{n}}\right)\cdot P = 1\cdot\underbrace{\left(\tfrac{1}{\sqrt{n}}\cdots\tfrac{1}{\sqrt{n}}\right)}_{\mathcal{L}_2\text{ norm}=1 \;\Rightarrow\; \text{normal}}$$

so this gets used a lot!

Important Theorem

As in Lake Wobegon, "where the women are strong, the men are good looking and all the children are above average", all theorems in this class are IMPORTANT!!

Thm Transition matrix P real & symmetric

$\Rightarrow \exists$ e-vecs $v^{(1)} \ldots v^{(n)}$

forming orthonormal basis with corresponding e-values $1=\lambda_1 \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$

& $v^{(1)} = \frac{1}{\sqrt{n}}(1 \ldots 1)$

chosen so that $\|v^{(1)}\|_2 = 1$

(won't prove here)

Useful Facts:

Assume P has all positive entries

& e-vecs $v^{(1)} \dots v^{(n)}$ with

corresponding e-vals $\lambda_1 \cdots \lambda_n$

Facts

(1) αP has e-vecs $v^{(1)} \dots v^{(n)}$ with corresponding evals $\alpha\lambda_1, \dots, \alpha\lambda_n$

(2) $P+I$ " " " " " " $\lambda_1+1, \dots, \lambda_n+1$

(3) P^k " " " " " " $\lambda_1^k, \dots, \lambda_n^k$

(4) P stochastic $\Rightarrow$ $|\lambda_i| \leq 1 \quad \forall i$

Why?

(1) $vP = \lambda v \iff v \cdot \alpha \cdot P = \lambda \cdot \alpha \cdot P$

(2) $v(P+I) = vP + vI = \lambda v + v = (\lambda+1)v$

Note: add self-loops: $\frac{P+I}{2}$ = "stay put with prob $1/2$ & walk with prob $1/2$"

$\Rightarrow$ new eigenvalues $\frac{\lambda_1+1}{2}, \dots, \frac{\lambda_n+1}{2}$

(3) $vP^k = (vP)P^{k-1} = \lambda v P^{k-1} = \lambda^2 v P^{k-2} = \dots = \lambda^k v$

k-step walks

(4) $\forall i$, let $I = \{ j \mid v_j^{(i)} > 0 \}$

then $$\lambda \sum_{j \in I} v_j^{(i)} = \sum_{j\in I} \overbrace{\sum_k v_k^{(i)} P_{kj}}^{\text{computes } j\text{th entry of } v\cdot P}$$

$$\le \sum_{\substack{j,k \\ \text{s.t. } j,k \in I}} v_k^{(i)} P_{kj}$$ since entries of v not in I are ≤ 0 (& P_{kj} is always ≥ 0)

$$\le \sum_{k \in I} v_k^{(i)} \underbrace{\sum_{j \in I} P_{ij}}_{\le 1 \text{ since stochastic}} \le \sum_{k\in I} v_k^{(i)}$$

$\Longrightarrow \lambda \le 1$

Note if $v^{(1)} \dots v^{(n)}$ orthonormal basis then

any vector w is expressible as linear combination of $v^{(i)}$'s

$$w = \sum \alpha_i v^{(i)}$$

& L_2-norm of w is $\underbrace{\sqrt{\sum \alpha_i^2}}_{\text{why?}}$

$$\|w\|_2 = \sqrt{\sum \alpha_i v^{(i)} \cdot \sum \alpha_j v^{(j)}}$$

$$= \sqrt{\sum_{i,j} \alpha_i \alpha_j \underbrace{v^{(i)} v^{(j)}}_{\substack{=0 \text{ if } i \neq j \\ =1 \text{ if } i = j}}}$$

$$= \sqrt{\sum_i \alpha_i^2}$$

(*)
will use
this soon

Recall:

Stationary distribution:

π s.t. $\pi = \pi P$

(taking more steps in r.w. keeps you
in same distribution)

recall: P ergodic $\Rightarrow$ π exists & unique

(for graphs, can always take $\frac{P+I}{2}$)

Mixing Times

How long does it take to reach stationary distribution?

def. $\varepsilon > 0$

Mixing time, $T(\varepsilon)$, of M.C. A with stationary dist π is min t s.t.

$\forall \pi^{(0)}$, $\quad \|\pi - \pi^{(0)} A^t\|_1 < \varepsilon$

def. M.C. A is rapidly mixing if

$$T(\varepsilon) = \mathrm{poly}(\log \underset{\uparrow \\ \#\text{states}}{n}, \log 1/\varepsilon)$$

examples: r.w. on complete graph, random graph

note that mixing time of $\frac{P+I}{2}$ is at most 2x more

<u>Thm</u> P is transition matrix of undirected,

can put self loop on each node → non k-partite, d-reg connected graph

π_0 is start dist.

π is stationary dist $= (\frac{1}{n}, \dots, \frac{1}{n})$

so $\pi P = P$

Then $\|\pi_0 P^t - \pi\|_2 \leq |\lambda_2|^t$

exponentially decreasing dist if $1-\lambda_2$ is const!!

<u>Proof</u>

P real, Symmetric $\Rightarrow$

evecs $v^{(1)} \dots v^{(n)}$ are orthonormal basis

with e-vals $1 = \lambda_1 \geq |\lambda_2| \geq \dots \geq |\lambda_n|$