Today

Random walks

Stationary Distributions

Cover Times

UST Conn

Aperiodic:

$\forall x \quad \gcd \{ t : P^t(x,x) > 0 \} = 1$ ← gcd of "possible" cycle length = 1

<u>not</u> bipartite, k-partite...

<u>Thm</u> Ergodic $\Leftrightarrow$ Irreducible + Aperiodic

## Stationary Distributions

does it depend on $\pi_0$? $\Big\{$ stationary distribution $\pi$

$\forall y \quad \pi(y) = \sum_x \pi(x) P(x,y)$ $\Big\}$ so $\pi^{(t)} = \pi^{(t-1)}$

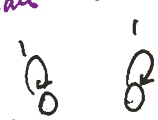
Will consider $P$ s.t. $\pi$ is unique & exists $\Big\}$ i.e. doesn't depend on $\pi_0$

if periodic: could have no stat. dist. or several

if $\pi_0 = (0,1)$ then $\pi_{2i} = (0,1)$

$\pi_{2i+1} = (1,0)$

if reducible: could have lots of stat. dist.



Some stat dists:

$(\frac{1}{2}, \frac{1}{2})$ $(0,1)$ $(1,0)$...

<u>Important Thm</u> every ergodic M.C. has unique stationary distribution

Stationary dist. of undirected graph:

$$\pi = \left( \frac{\deg(x_1)}{2|E|}, \frac{\deg(x_2)}{2|E|}, \dots \right)$$

- so d-regular graphs have $\pi$ = uniform
  (also indegree = outdegree = d digraphs
  + $\underbrace{\text{doubly stochastic } P}_{\text{this implies the others!}}$ M.C.'s)
- not true in general for digraphs
- bipartite, periodic graphs may have other stat. dists.

## Hitting times

def. $h_{ii}$ = E[time starting at i to return to i]

Thm $h_{ii} = \frac{1}{\pi(i)}$ ← Very useful theorem!

def. $h_{ij}$ = E[time starting at i to reach j]

## Cover time of undirected graph

$C_u(G)$ = E[# steps to reach all nodes in G on walk starting from u]

$C(G) = \max_u C_u(G)$

↑ worst start pt

## Cover time Examples:

- $\mathcal{C}(K_n^*)$ where $K_n^*$ = complete graph with self-loops at each node

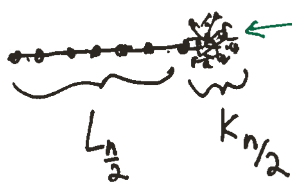
  $= \Theta(n \ln n)$ by coupon collector argument

- $\mathcal{C}(L_n^*)$ where $L_n^*$ = n node line with self-loops at each node ↖ so aperiodic

  $= \Theta(n^2)$

- $\mathcal{C}$(lollipop)

  $= \Theta(n^3)$




← worst place to start

**Thm** $\mathcal{C}(G) \leq 8m(n-1)$

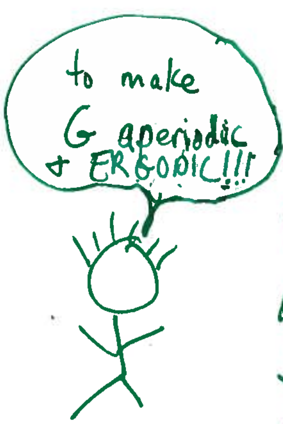
**Proof**

First - transform $G$ (undirected) into $G'$ (directed) (see example on pg 8)

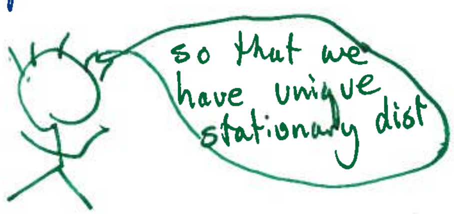
to make $G$ aperiodic, add $d$ self loops to each $u$ (i.e. take self-loop with prob $1/2$)


**Claim:** $\mathcal{C}(G') = 2\,\mathcal{C}(G)$

why? can transform paths in $G'$ by removing self-loops, expected # self-loops = $1/2$ (length of path)







Next, commute times + a lemma:

def. $C_{ij}$ = E[#steps for r.w. starting at $i$ to hit $j$ + return to $i$] "Commute time"

Claim $C_{ij} = h_{ij} + h_{ji}$ (linearity of expectation)

Lemma $\forall (u,v) \in E \quad C_{uv} \le O(m)$

Pf of lemma

Key idea: (actually will show $C_{vu} \le O(m)$ but it's symmetric)
if traverse $(u,v)$ twice

$(u,v) \ldots (u,v)$

have performed commute from $v \rightsquigarrow u \rightsquigarrow v$

very specific kind of commute from $u \to v \to u$ happens often enough

Plan: show E[time between visits to $(u,v)$] is $O(m)$
$\Longrightarrow C_{uv}$ is $O(m)$

Given $G' = (V, E')$ ($G$ with added self loops)
Construct (directed) $G''$ representing walks on directed edges of $G'$

line graph

$E' \to V''$ new nodes are edges $(u,v)$ in $G'$
$\underbrace{(u,v)(v,w)}_{\text{consecutive edges}} \to E''$ new edges are length 2 paths in $G'$

visit edge in $G'$ twice $\Leftrightarrow$ visit node in $G''$ twice example

$G$


$\Longrightarrow$

$G'$


$1 \to 2 \to 1$

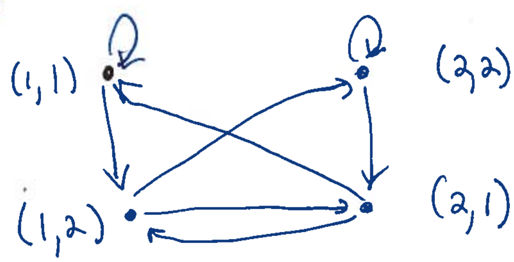
$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 1 \\ 2 & 1 & 0 \end{array}$$

$1 \to 1 \to 2 \to 2 \to 2 \to 1$

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1/2 & 1/2 \\ 2 & 1/2 & 1/2 \end{array}$$

$\Downarrow$

$G''$

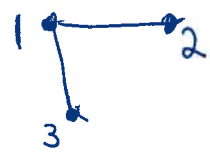



| | (1,1) | (1,2) | (2,2) | (2,1) |
|---|---|---|---|---|
| (1,1) | 1/2 | 1/2 | 0 | 0 |
| (1,2) | 0 | 0 | 1/2 | 1/2 |
| (2,2) | 0 | 0 | 1/2 | 1/2 |
| (2,1) | 1/2 | 1/2 | 0 | 0 |

(more complicated example) rw⑧

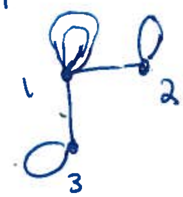
example

G




$\Rightarrow$

G'




$$\begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & 0 & 1/2 & 1/2 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ \hline \end{array}$$

$1 \to 2 \to 1$

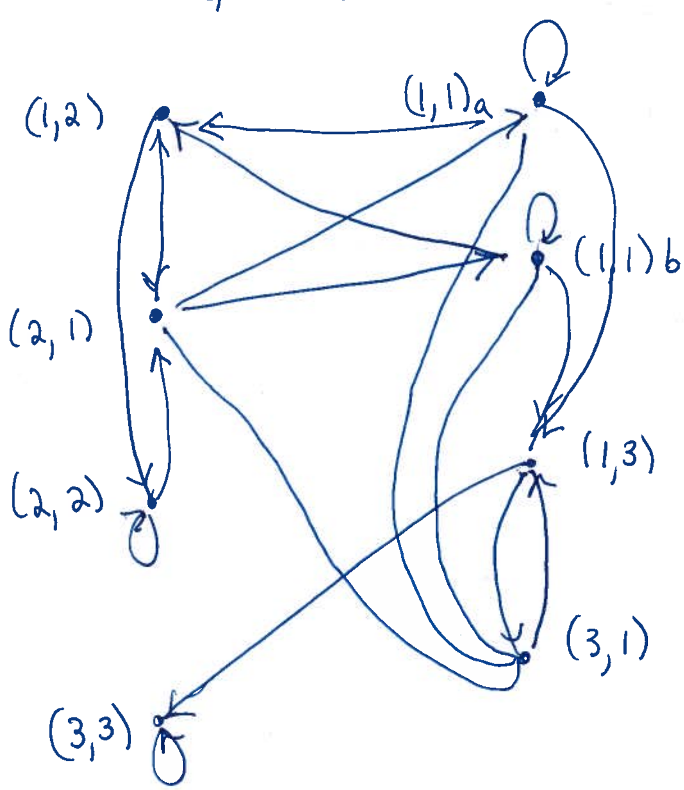
$$\begin{array}{c|ccc|} & 1 & 2 & 3 \\ \hline 1 & 1/2 & 1/4 & 1/4 \\ 2 & 1/2 & 1/2 & 0 \\ 3 & 1/2 & 0 & 1/2 \\ \hline \end{array}$$

$1 \to 1 \to 2 \to 1$

$\Downarrow$

$(1,1) \to (1,2) \to (2,1)$

G''



Note: $G''$ is doubly stochastic:

note that $G''$ is directed but has underlying structure from undirected graph

why? $Q_{(u,v)(v,w)} = P_{vw} = \frac{1}{d(v)}$ if $(u,v)\,(v,w) \in E$

$\forall\ (v,w) \in \underset{V''}{\overset{\shortparallel}{E}}$ $\quad \underbrace{\sum_{\substack{(u,v) \text{ s.t.} \\ (u,v)(v,w) \in E'}}}_{\text{column sum}} Q_{(u,v)(v,w)} = \sum_{u \text{ s.t. } (u,v) \in E} \frac{1}{d(v)} = 1$

$\therefore\ \pi$ of $G''$ is uniform

we need that walk on $G''$ is ergodic. Irreducible follows from $G'$ irreducible. Aperiodic comes from self-loops.

so $\underset{\substack{\uparrow \\ \text{edge in } G'}}{\pi_u} = \frac{1}{|V''|} = \frac{1}{4m}$ $\leftarrow$ # edges in $G'$ ($(u,w) \to (u,v), (v,u)$ + 2 self loops)

$\underset{\substack{\uparrow \\ \text{edge in } G'}}{h_{uu}} = \frac{1}{\pi_u} = 4m$ for all nodes $u$ in $G''$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad (a,b)$ in $G'$

$\blacksquare$

if start at $v$ conditioned on coming from edge $(u,v)$ (proof of lemma)
expect $\leq 4m$ steps to see $(u,v)$ again.
But, its an M.C! so conditioning doesn't affect.
$\Rightarrow$ if start at $v$, expect to see $(u,v)$ in $\leq 4m$ steps.
$\Rightarrow$ $C(v,u) = C(u,v) = O(m)$

Note: valid only for $(u,v) \in E$

$\blacksquare$

## Wrapping it up:

**Lemma** $C(G) = O(nm) = O(n^3)$

**Pf.**

start vertex $v_0$

$T \leftarrow$ (any) spanning tree rooted at $v_0$

# edges in $T = n-1$

$v_0 \; v_1 \cdots v_{2n-2}$ is depth 1st traversal
$\quad\quad\quad\quad\; \rotatebox{90}{=}$
$\quad\quad\quad\quad\; v_0$

s.t. each edge appears twice, once in each direction $(a,b)$ & $(b,a)$

$$C(G) \le \sum_{j=0}^{2n-3} h_{v_j v_{j+1}}$$

$$= \sum_{(u,v)\in T} C_{uv}$$ since $C_{uv} = h_{uv} + h_{vu}$

$$= O\Big(\sum_{(u,v)\in T} m\Big)$$

$$= O(nm)$$

$\blacksquare$

## S-T connectivity (UST-Conn)

Input: Undirected $G$, nodes $s, t$

Output: "Yes" if $s$ & $t$ connected

"No" o.w.

Can solve in poly time, in many ways.

What about small space?

RL = class of problems solvable by randomized log-space computations

[no charge for input space (read only), but can only have const # ptrs ...]

**Thm** UST-Conn $\in$ RL

Algorithm:

start at $s$

take random walk for $cn^3$ steps

if ever see $t$, output "Yes"

o.w. output "No"

Complexity:

Keep track of # steps so far — log space

\# edges at each node & toss coin to pick one randomly

Behavior:

If $s,t$ not connected, never output "yes"

If $s,t$ connected

$$h_{st} \le C_s(G_s) \le n^3$$

connected component of s

$$\Pr[\text{output "no"}] = \Pr[\text{start at } s, \text{ walk} \ge c \cdot E[C_s(G_s)] \text{ steps} + \text{don't see } t]$$

$$\le \frac{1}{c} \quad \text{by Markov's} \neq \quad \blacksquare$$

Comments

- Actually USTCONN $\in L$ !!!

- Open is $RL = L$ ?

  we know $RL \in L^{3/2}$