Today:

Weak learning of monotone fctns.

<u>def</u> partial order $\preceq$

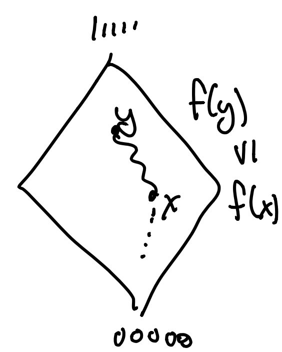
$$x \leq y \quad \text{iff} \quad \forall i \;\; x_i \leq y_i$$

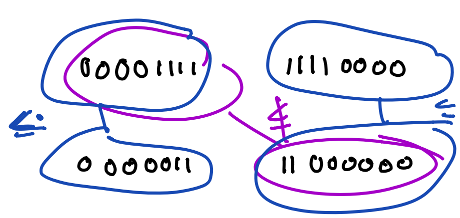
$$0011010 \leq 0111011$$

$$\not\leq 001\cancel{0}111$$

monotone fnctn f

$$x \leq y \Rightarrow f(x) \leq f(y)$$





Can you learn the class of monotone fctns?

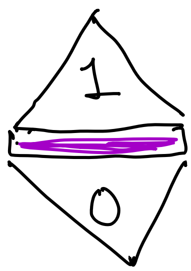
In h.w. $2^{O(\sqrt{n})}$ random samples suffice for uniform dist

Arbitrary distributions hard, even with queries

Occam $\Rightarrow$ # monotone fctns $\approx 2^{2^n/\sqrt{n}}$

so $\frac{2^n}{\sqrt{n}}$ samples suffice

consider "slice fctns"



} #1's > #0's

$\leftarrow$ row where #1's = #0's

} #0's > #1's

can be anything $\rightarrow$

$\approx \frac{2^n}{\sqrt{n}}$ elts in middle row

$2^{2^n/\sqrt{n}}$ possible fctns

a "hard" distribution:
uniform on middle row

any learning alg needs to see most of middle row (purple)

$\Rightarrow \Omega(2^n/\sqrt{n})$ queries/samples in PAC model

<u>Today</u> uniform on $\{0,1\}^n$ (whole hypercube) with queries

can get <u>slight</u> win!

Can weakly learn:

all monotone fctns have weak agreement with some dictator fctn,

$\{x_1, x_2, \dots x_n\}$ $\quad f(x) = x_i$

(switch to $f: \{\pm 1\}^n \to \{\pm 1\}$)

Thm $\forall f$ monotone, $\exists\, g \in \underbrace{\{\pm 1, x_1, x_2, \dots x_n\}}_{S}$

s.t. $\Pr_x[f(x) = g(x)] \geq \frac{1}{2} + \Omega(\frac{1}{n})$

If true $\Rightarrow$

gives alg for weak "learning" of monotone f

with agreement $\frac{1}{2} + \Omega(\frac{1}{n})$

by testing all fctn in S

& outputting any one that agrees

$\geq \frac{1}{2} + \Omega(\frac{1}{n})$

Pf of thm

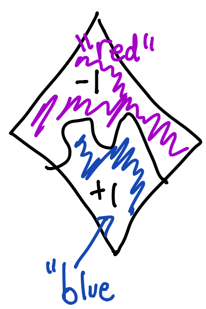
Case 1 $\quad f(x)$ has $\frac{3}{4}$-agreement with $+1$ or -1

Case 2 $\quad \Pr[f(x)=1] \in [\frac{1}{4}, \frac{3}{4}]$

"balanced"

first a "break":

define "Influence"



$\#$ nodes $= 2^n$

$\#$ edges $= \frac{n \cdot 2^n}{2}$

each level has $\binom{n}{j}$ wt j nodes

monotone $\Rightarrow$ no blue above any red

$$\mathrm{Inf}(f) \equiv \frac{\#\text{red-blue edges}}{2^{n-1}} = \Pr_x[f(x) \neq f(x^{\oplus i})]$$

x with ith bit flipped

$(b_1 b_2 \cdots b_{i-1} \, 1 \, b_{i+1} \cdots b_n)$

$(b_1 b_2 \cdots b_{i-1} \, 0 \, b_{i+1} \cdots b_n)$

$$\mathrm{Inf}_i(f) \equiv \frac{\#\text{ red-blue edges in } i\text{th dir}}{2^{n-1}}$$

earlier lecture

On h.w.: <u>Thm</u> for monotone f, $\mathrm{Inf}_i(f) = \hat{f}(\{i\}) \equiv 2 \cdot \Pr[f(x) = x_i] - 1$

$$\chi_{\{i\}} = \prod_{j \in \{i\}} x_j = x_i$$

<u>Plan</u> Show $\exists i \; \mathrm{Inf}_i(f) \geq \Omega(\frac{1}{n})$

$$\Rightarrow \Pr[f(x) = x_i] \geq \frac{1}{2} + \frac{\mathrm{Inf}_i(f)}{2}$$

$$\geq \frac{1}{2} + \Omega\left(\frac{1}{n}\right)$$

Important tool:

Canonical Path Argument

<u>Plan</u>

1) define canonical path for every red-blue pair of nodes
   - (# red nodes) x (# blue nodes) such paths
   - must cross at least one red-blue edge

2) show upper bnd on # of c.p.'s passing thru <u>any</u> edge

3) conclude lower bnd on # of red-blue edges

Part 1):

$\forall (x,y)$ s.t. x is red (but not necessarily
y is blue $x \le y$ or $y \le x$)

"canonical path" from x to y is:

scan bits left to rt, flipping where needed

each flip $\rightsquigarrow$ step in path

example:

	1	2	3	4	
$x =$	-1	$+1$	$+1$	$+1$	x
					$\downarrow$
$w =$	$+1$	$+1$	$+1$	$+1$	w
					$\downarrow$
$z =$	$+1$	-1	$+1$	$+1$	z
					$\downarrow$
$y =$	$+1$	-1	$+1$	-1	y

<u>note</u> path can go up/down as much as it wants

how many red-blue x, y pairs?

$$\Pr[f(x)=1] \in [\tfrac{1}{4}, \tfrac{3}{4}]$$

$$\# \text{ paths} \geq \underbrace{\frac{1}{4}\cdot 2^n}_{\text{l.b. on } \#\text{red}} \cdot \underbrace{\frac{1}{4}\cdot 2^n}_{\text{l.b. on } \#\text{blue}} = \frac{1}{16}\cdot 2^{2n}$$

Part 2 of plan:

For any (red-blue) edge e,
how many x-y pairs can cross it
with canonical x-y path?

x: $x_1 x_2 \cdots \quad x_n$

$e = (u, u^{\oplus i})$

u: $y_1\, y_2 \cdots y_{i-1}\, x_i\, x_{i+1} \cdots x_n$

$u^{\oplus i}$: $y_1\, y_2 \cdots y_{i-1}\, y_i\, x_{i+1} \cdots x_n$

y: $y_1 y_2 \quad \cdots \quad y_n$

$$\underbrace{y_1 \cdots y_{i-1}}_{2^{i-1}} x_i\, x_{i+1} \cdots x_n$$

2^{i-1} possible x's could reach here

$$y_1 \cdots y_{i-1}\, y_i \underbrace{x_{i+1} \cdots x_n}_{2^{n-i}}$$

2^{n-i} possible y's could be reached

$$\leq (2^{i-1}) \cdot 2^{n-i} = 2^n$$

total settings of prefix of x & suffix of y consistent with e

Part 3 of plan

$$(\#\text{ red-blue edges}) \times (\underbrace{\text{max } \#\text{ canonical paths}}^{\le 2^n} \text{ that can use it})$$

$$\ge \underbrace{\#\text{ red-blue canonical paths}}_{\ge \frac{1}{16} \cdot 2^{2n}}$$

$\uparrow$ since each red-blue c.p. used a red-blue edge

So $\#$ red-blue edges $\ge \dfrac{\frac{1}{16} 2^{2n}}{2^n} = \frac{1}{16} 2^n$

$= \Omega(2^n)$

So $\exists\, i$ st $\ge \dfrac{2^n}{16} \cdot \dfrac{1}{n}$ red-blue edges in dir i

So $$\mathrm{Inf}_i(f) \geq \frac{\frac{2^n}{16} \cdot \frac{1}{n}}{2^{n-1}} = \frac{1}{8n} = \hat{f}(\{i\})$$

$$= 2 \cdot \Pr[f(x) = x_i] - 1$$

$$\Rightarrow \Pr[f(x) = x_i] \geq \frac{1}{2} + \frac{1}{16 \cdot n}$$

$\blacksquare$