Today:

- more Fourier analysis
- finish linearity testing
- begin learning

From last time:

- def. f is "linear" if $\forall x, y$ $f(x) + f(y) = f(x+y)$

- change $\{0,1\}^n$ to $\{+1,-1\}^n$

$+$	0	1
0	0	1
1	1	0

$+$ to $\times$

$\times$	+1	-1
+1	+1	-1
-1	-1	+1

$f(x) + f(y) = f(x+y)$ to $f(x) \cdot f(y) = f(x \odot y)$

coordinatewise mult

- define $\delta_f \equiv \Pr_{x,y}\left[f(x) \cdot f(y) \neq f(x \odot y)\right]$

"rejection probability"

- Thm $\delta_f = \mathbb{E}_{x,y}\left[\dfrac{1 - f(x)f(y)f(x \odot y)}{2}\right]$ for $f : \{\pm 1\}^n \to \{\pm 1\}$

note: $\chi_\emptyset$ is all 1's fcfn

- $\forall S \subseteq \{1..n\}$

$$\chi_S(x) = \prod_{i \in S} x_i \qquad \text{Parity fcfns}$$

- Thm $\{\chi_S\}$ form an orthonormal basis with respect to inner prod

$$\langle f, g \rangle = \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} f(x)\, g(x)$$

$\Rightarrow$ all $f : \{\pm 1\}^n \to \{\pm 1\}$ can be expressed as lin comb of χ_S's

---

Today:

$$\hat{f}(S) \equiv \langle f, \chi_S \rangle = \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} f(x) \cdot \chi_S(x)$$

"Fourier Coefficients" of f

Thm $\forall f \qquad f(x) = \sum_S \hat{f}(S) \cdot \chi_S(x)$

Fourier coefficients of linear fctns:

Fact $\underset{\substack{\parallel\\ f(x)=\chi_S(x)}}{f}$ linear $\Longleftrightarrow$ $\exists\, S \subseteq [n]$ s.t. $\hat{f}(S)=1$ (↑ one is big)

$\quad + \forall\, T \neq S \quad \underbrace{\hat{f}(T)=0}_{\text{others } 0}$

Fourier coeffs characterize distance to linear:

Lemma $\forall\, S \subseteq [n]$

$$\hat{f}(S) = 1 - 2\,\underbrace{\Pr_x[f(x) \neq \chi_S(x)]}_{dist(f,\chi_S)} = 2\Pr_x[f(x)=\chi_S(x)]-1$$

Pf.

$$2^n \hat{f}(S) = \sum_x f(x)\chi_S(x)$$

$$= \sum_{\substack{x \text{ s.t.}\\ f(x)=\chi_S(x)}} 1 \; + \sum_{\substack{x \text{ s.t.}\\ f(x)\neq\chi_S(x)}} -1$$

$$= 2^n\cdot(1-dist(f,\chi_S)) + 2^n \cdot dist(f,\chi_S)\cdot(-1)$$

$$= 2^n(1 - 2\,dist(f,\chi_S))$$

∎

<u>Observation</u> 2 distinct linear fctns (mapping $\{\pm 1\}^n \to \pm 1$)

differ on exactly $\frac{1}{2}$ of pts

<u>Pf.</u>

$f = \chi_T \qquad T \neq S$

$g = \chi_S \qquad \hat{f}(S)$ by def

$$0 = \langle \chi_T, \chi_S \rangle = 1 - 2\,\text{dist}[\overset{f}{\chi_T(x)}, \overset{g}{\chi_S(x)}]$$

last time

$$\Downarrow$$

$$\text{dist}[\chi_T, \chi_S] = \frac{1}{2} \quad \blacksquare$$

<u>Corr</u> Fraction of inputs on which $\chi_S = +1$?

if $S = \emptyset$: $E_x[\chi_S(x)] = 1$

if $S \neq \emptyset$: $E_x[\chi_S(x)] = ?$

$\chi_\emptyset \equiv 1$ on all inputs

χ_S + $\chi_\emptyset$ agree on $\frac{1}{2}$ inputs

$\Rightarrow \chi_S = 1$ on $\frac{1}{2}$ inputs

$\qquad -1$ on $\frac{1}{2}$ inputs

$\Rightarrow E_x[\chi_S(x)] = 0$

<u>example</u> f = all (-1)'s

$f = -\chi_\phi$

what is $\hat{f}(\phi)$? $\mathrm{dist}(f, \chi_\phi) = 1$

$\Rightarrow \hat{f}(\phi) = -1$

what is $\hat{f}(S)$ for $S \neq \phi$?

$\chi_S(x) = +1$ on exactly $\frac{1}{2}$ x's $\leftarrow$ dis agree

-1 " " " " $\leftarrow$ agree

$\Rightarrow$ f + $\chi_S(x)$ agree on exactly $\frac{1}{2}$ x's

$\Rightarrow$ $\mathrm{dist}(f, \chi_S) = \frac{1}{2}$

$\hat{f}(S) = 0$

Useful tool:

Plancherel's Identity:

$$\langle f, g\rangle = \langle \sum_{S\subseteq[n]} \hat{f}(S)\chi_S, \sum_{T\subseteq[n]} \hat{g}(T)\chi_T \rangle$$

$$= \sum_{S,T} \hat{f}(S)\cdot\hat{g}(T) \underbrace{\langle \chi_S, \chi_T\rangle}_{\substack{=0 \text{ if } S\neq T \\ =1 \text{ if } S=T}}$$

bilinearity of $\langle \cdot, \cdot \rangle$

$$= \sum_S \hat{f}(S)\,\hat{g}(S)$$

Parseval's Identity:

$$\forall f \quad \langle f, f\rangle = \sum_S \hat{f}(S)^2$$

"Boolean" Parseval's:

$\forall f$ boolean (range $= \{+1,-1\}$) $\quad \langle f, f\rangle = \frac{1}{2^n}\sum_x \underbrace{f(x)\cdot f(x)}_{\substack{+1 \\ \text{since } f \\ \text{Boolean}}} = 1$

$$\Rightarrow \sum_S \hat{f}(S)^2 = 1$$

Back to proof of linearity test:

Recall: • given f s.t.

$$\delta_f \equiv \Pr[f(x \odot y) \neq f(x) \cdot f(y)]$$

$$= E\left[\frac{1 - f(x) f(y) f(x \odot y)}{2}\right]$$

• f is ε-close to linear if $\exists$ g linear s.t. $f + g$ agree on $\geq 1-\varepsilon$ fraction of inputs

<u>Thm</u> f is δ_f-close to some linear fctn

<u>Pf</u>.

$$E_{x,y}[f(x) f(y) f(x \odot y)]$$

$$= E_{xy}\left[\left(\sum_S \hat{f}(S) \chi_S(x)\right) \cdot \left(\sum_T \hat{f}(T) \chi_T(y)\right) \cdot \left(\sum_U \hat{f}(U) \chi_U(x \odot y)\right)\right]$$

$$= E_{xy}\left[\sum_{S,T,U} \hat{f}(S)\cdot\hat{f}(T)\hat{f}(U)\,\chi_S(x)\chi_T(y)\,\chi_U(x\odot y)\right]$$

$$= \sum_{S,T,U} \hat{f}(S)\hat{f}(T)\hat{f}(U)\,\underbrace{E_{x,y}\left[\chi_S(x)\chi_T(y)\chi_U(x\odot y)\right]}$$

<u>note</u>: 1) if $S = T = U$

$$\chi_S(x)\cdot\chi_{\cancel{T}\,S}(y)\chi_{\cancel{U}\,S}(x\odot y) = \prod_{i\in S} x_i \cdot \prod_{i\in S} y_i \cdot \prod_{i\in S} \overset{x_i\cdot y_i}{\overset{\downarrow}{(x\odot y)_i}}$$

$$= \prod_{i\in S} \underbrace{x_i^2}_{+1}\cdot \underbrace{y_i^2}_{+1} = 1$$

2) if $\neg(S = T = U)$

$$E_{x,y}\left[\chi_S(x)\chi_T(y)\chi_U(x\odot y)\right]$$

$$= E_{xy}\left[\prod_{i\in S} x_i \prod_{j\in T} y_j \prod_{k\in U} x_k\cdot y_k\right]$$

$$= E_{xy}\left[\prod_{i\in S\Delta U} x_i \prod_{j\in T\Delta U} y_j\right]$$

$$= \underbrace{E_x\left[\prod_{i\in S\Delta U} x_i\right]}\cdot E_y\left[\prod_{j\in T\Delta U} y_j\right]$$ since x,y indep

if $S \neq U$

then $S \Delta U \neq \emptyset$

so $E_x[\prod_{i \in S \Delta U} x_i] = 0$

if $T \neq U$

then

$= 0$

by assumption $\neg(S = T = U)$

either $S \neq U$ or $T \neq U$

So sum $= 0$

$= 0$

So only $S = T = U$ terms remain

$$E_{xy}[f(x) f(y) f(x \odot y)]$$

$$= \sum_{S=T=U} \hat{f}(S)^3$$

$$\leq \left(\max_S \hat{f}(S)\right) \cdot \underbrace{\sum \hat{f}(S)^2}_{=1}$$ by Boolean Parseval's

$$= \max_S \hat{f}(S)$$

$$= \max_S \left(1 - 2\,\mathrm{dist}(f, \chi_S)\right)$$

$$= 1 - 2 \min_{S} \text{dist}(f, \chi_S)$$

so $$\delta_f = \frac{\cancel{1} + (\cancel{1} + \cancel{2} \min_{S} \text{dist}(f, \chi_S))}{\cancel{2}}$$

$$= \min_{S} \text{dist}(f, \chi_S)$$

∎.