6.842 Randomness and Computation	April 23, 2020
Homework 6	
Lecturer: Ronitt Rubinfeld	*Due Date: May 5, 2020*

Influence of Variables on Functions

For $x = (x_1, \ldots, x_n) \in \{\pm 1\}^n$, let $x^{\oplus i}$ be x with the i-th bit flipped, that is,

$$x^{\oplus i} = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n).$$

The *influence of the* i*-th variable on* $f : \{\pm 1\}^n \to \{\pm 1\}$ is

$$\mathrm{Inf}_i(f) = \Pr_x \left[f(x) \neq f\left(x^{\oplus i}\right)\right].$$

The *total influence* of f is

$$\mathrm{Inf}(f) = \sum_{i=1}^{n} \mathrm{Inf}_i(f).$$

This definition will be useful for the last two problems.

Turn in solutions to **TWO out of the following THREE** problems:

1. Show that if there is a PAC learning algorithm for a class C with sample complexity $\mathrm{poly}(\log n, 1/\epsilon, 1/\delta)$, then there is a PAC learning algorithm for C with sample complexity dependence on δ (the confidence parameter) that is only $\log 1/\delta$ – i.e., the "new" PAC algorithm should have sample complexity $\mathrm{poly}(\log n, 1/\epsilon, \log 1/\delta)$. (It is ok to assume that the learning algorithm is over the uniform distribution on inputs, although the claim is true in general.)

2. A function $f : \{\pm 1\}^n \to \{\pm 1\}$ is *monotone* if, for all $x, y \in \{\pm 1\}^n$ such that $x \preceq y$ (meaning $x_i \leq y_i \ \forall i \in [n]$), we have $f(x) \leq f(y)$.

   (a) Show that for any monotone function $f : \{\pm 1\}^n \to \{\pm 1\}$, the influence of the i^{th} variable is equal to the value of the Fourier coefficient of $\{i\}$, that is $Inf_i(f) = \hat{f}(\{i\})$.

   (b) Show that the majority function $f(x) = \mathrm{sign}(\sum_i x_i)$ maximizes the total influence among n-variable monotone functions mapping $\{\pm 1\}^n$ to $\{\pm 1\}$, for n odd.

3. Consider the sample complexity required to learn the class of monotone functions mapping $\{+1, -1\}^n$ to $\{+1, -1\}$ over the uniform distribution (without queries).

   (a) Show that
   $$\sum_{|S| \geq Inf(f)/\epsilon} \hat{f}(S)^2 \leq \epsilon$$
   .

   (b) Show that the class of monotone functions can be learned to accuracy ϵ with $n^{\Theta(\sqrt{n}/\epsilon)} = 2^{\tilde{O}(\sqrt{n}/\epsilon)}$ samples under the uniform distribution (where the confidence parameter δ is some small constant).
   Hint: Use the result from the previous problem.