Lecture 18:

Local Computation Algorithms for MIS

Maximal Independent Set:

def. $U \subseteq V$ is "Maximal Independent Set" (MIS)

if (1) $\forall$ $u_1, u_2 \in U$ $(u_1, u_2) \notin E$ (independent)

(2) $\nexists$ $w \notin U$ s.t. $U \cup \{w\}$ is also independent (maximal)

Today's assumption (important):

G has max degree d

Note: Maximum Independent set is NP-complete

Maximal " " can be solved by greedy

Distributed Algorithm for Maximal Independent Set (MIS):

"Luby's Algorithm" (one of many variants)

- all nodes set to "live"
- repeat K times in parallel:

  $\forall$ nodes v, v "selects" self with prob $= \frac{1}{2d}$

  if v live: if (v selects self) + (no nbr w of v selects itself)

  then
  (1) v added to MIS
  (2) v + nbrs of v removed from graph (set to "dead")

(for purposes of analyses, continue to select selves even after "die")

If goal is to "kill" the whole graph:

Thm. $\Pr[\#\text{ phases} \geq 8 \log n] \leq \frac{1}{n}$

Corr $E[\#\text{ phases}]$ is $O(d \log n)$ $\Leftarrow$ Can improve!!

Main Lemma $\Pr[v \text{ live} + \text{adds self to MIS in one round}] \geq \frac{1}{4d}$

Pf. $\forall v$ live:

$$\Pr[v \text{ selects self}] = \frac{1}{2d}$$

$$\Pr[\text{any } w \in N(v) \text{ selects self}] \leq \sum_{w \in N(v)} \frac{1}{2d} \quad \text{union bnd}$$

$$\leq \frac{d}{2d} = \frac{1}{2}$$

$$\therefore \Pr[v \text{ selects self} + \text{no nbr selects self}] \geq \frac{1}{2d}\left(1-\frac{1}{2}\right) = \frac{1}{4d}$$

$\Rightarrow$ **Corr.** $\Pr[v \text{ alive after } 4kd \text{ rounds}] \leq \left(1-\frac{1}{4d}\right)^{4kd} \leq e^{-k}$

Note Luby's alg uses $k = O(\log n)$

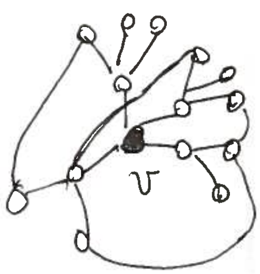
so union bnd $\Rightarrow$ all die

(also avoids dependence on d via smarter analysis)

Local Computation Algorithm for Luby's answer:

- Previous with $k = O(d \log d)$ gives:
  $O(d \log d)$ round distributed alg outputting
  one of { live – v alive after $\log d$ rounds
  in – v in MIS
  out – v not in MIS for sure

- Using "Parnas Ron" reduction: on input v



simulate v's view of computation in $\boxed{d^{O(\log d)} \text{ queries}}$ to input

degree

\# rounds

size of radius $O(\log d)$ ball around v

& output whether v is alive, in or out of MIS

Subroutine Lubystatus(v)

If v is in/out, we are done!
What if v is still alive?

To show: can still figure v out quickly given subroutine Lubystatus

LCA for MIS(v):

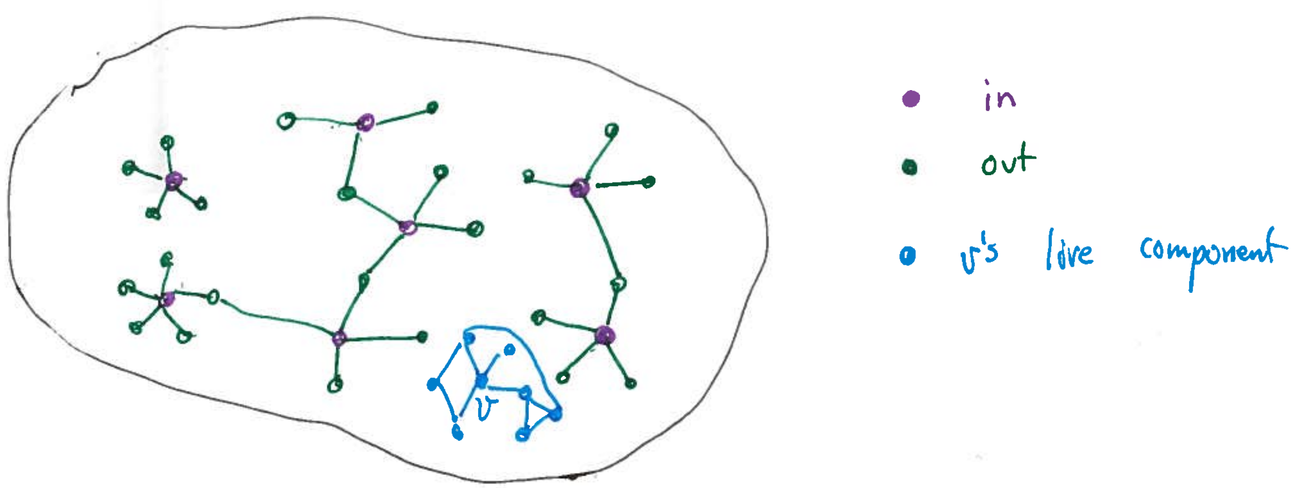
runtime

if Luby status(v) is in/out, output it & halt $\}$ $d^{O(d \log d)}$

else (1) do BFS to find v's connected component of live nodes $\}$ $d^{O(d \log d)}$ x size of component

(2) Compute lexicographically 1st MIS M' to that connected component

(3) output whether v in/out of M'



Runtime: need to bound size of live components

Bounding live component sizes:

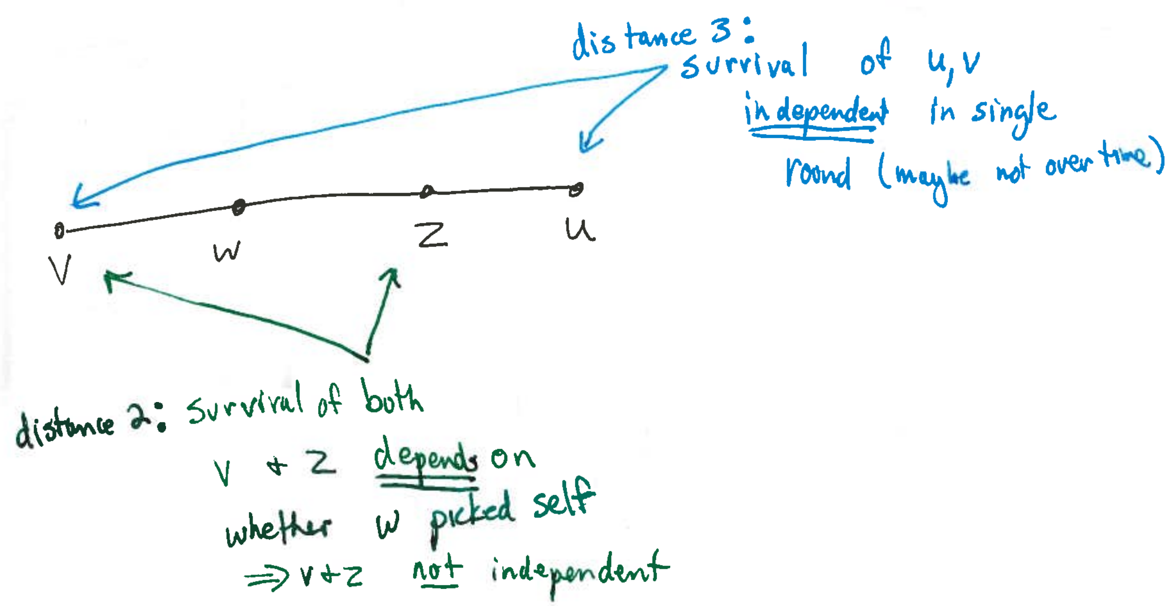
def. $A_v = \begin{cases} 1 & \text{if } v \text{ survives all rounds} \\ 0 & \text{o.w} \end{cases}$

$B_v = \begin{cases} 1 & \text{if } \nexists \text{ round s.t. } v \text{ picks self \& no } w \in N(v) \text{ picks self} \\ 0 & \text{o.w.} \end{cases}$

might not mean $v \in MIS$ if eg. v's nbr put in MIS in earlier round

Claim if v survives, $\nexists$ round s.t. v picks self & no $w \in N(v)$ picks self

independent for v & v' of distance ≥ 3



Corr to Claim $\forall$ W, if all nodes in W survive then $\nexists$ round s.t. any node v in W picks self & no $w \in N(v)$ picks self

Note: (1) survival of V can depend only on coin tosses of w's within distance 2 of V

$\Rightarrow \leq d^2$ other B_w's

Note (2):

Survival of v is "rare" over $c \cdot d \log d$ rounds

$$\Pr[\nexists \text{ round s.t. } v \text{ picks self} + \text{no } w \in N(v) \text{ picks self}]$$

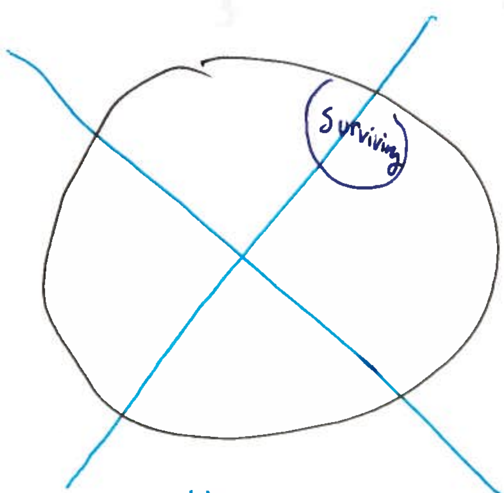
$$\leq \left(1 - \frac{1}{4d}\right)^{c \cdot d \log d} \qquad \text{picking } c \geq 20$$

$$\leq \frac{1}{8d^3}$$

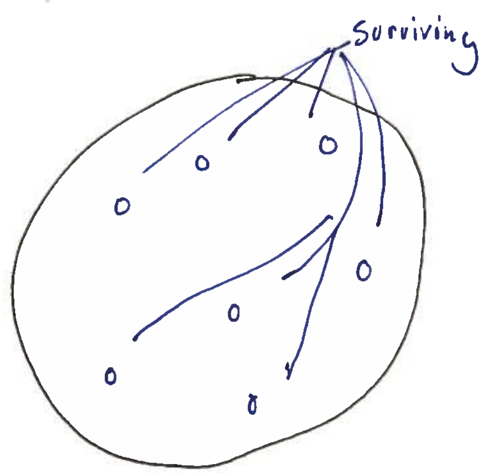
Notes (1) + (2) $\Rightarrow$

Survival "rare" + "$\approx$ independent"

$\Rightarrow$ good behavior: surviving nodes in small connected components

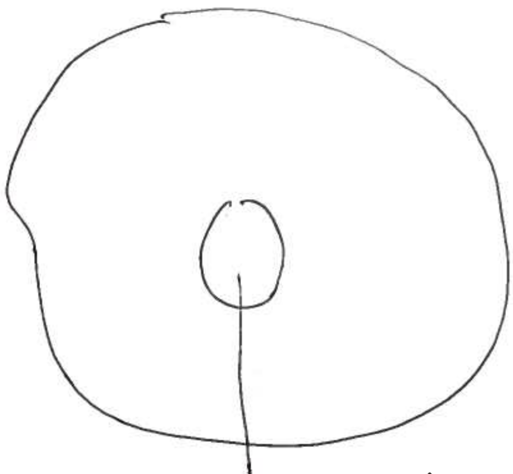


NO



MAYBE

Why can't we say anything about complete graph?



Component of size k survives

(1) $\binom{n}{k}$ Components $\Longleftarrow$ only $\approx n(4d^3)^k$ for $\deg \le d$ graphs

(2) survival within & without of component not independent

$\Uparrow$

lots of "independencies"

Claim After $O(d \log d)$ rounds, connected components of survivors are of size $\leq O(\text{polylog}(d) \cdot \log n)$

$\Rightarrow$ can use brute force!

Pf of Claim:

idea:

- any connected component that is large has lots of nodes that are independent (distance ≥ 3)
- these independent nodes are unlikely to simultaneously survive

do we need to "union bound" over all sets of size w? NO!!

Let $H^{(3)} \leftarrow$ graph s.t. nodes $\sim B_v$, edges $\sim B_v$ & B_w s.t. v & w are $\underbrace{\text{distance} = 3}_{\text{independent}}$ in G

$\deg(H^{(3)}) \leq d^3$

Observe: # connected components of size w $\leq$ # size w subtrees of $H^{(3)}$

why? map each connected component C to arbitrary spanning tree of C

mapping is 1-1

(note that each component could have <u>many</u> spanning trees)

Thm # size w trees in $H^{(3)} \le n(4d^3)^w$

Why?

nonisomorphic trees on w nodes $\le 4^w$

Process:

	# choices
• choose root	n
• choose tree	4^w
• choose placement w/in $H^{(3)}$	d^3 choices for 1st child
	" " " 2nd "
	⋮

total # choices: $n \cdot 4^w \cdot (d^3)^w$ ∎

note independent set I in $H^{(3)}$ of size w $\Rightarrow$ I's nodes $\ge$ dist 3 in G (pairwise)

$\Rightarrow \Pr[\text{ind set } I \text{ survives in } G] \le \left(\frac{1}{8d^3}\right)^{|I|}$

$\Rightarrow \Pr[\text{specific size } w \text{ tree survives in } H^{(3)}] \le \left(\frac{1}{8d^3}\right)^w$

$\Rightarrow \Pr[\exists \text{ size } w \text{ tree surviving in } H^{(3)}] \le \frac{n \cdot (4d^3)^w}{(8d^3)^w}$

$\Rightarrow$ for $w = \Theta(\log n)$,

$\Pr[\exists \text{ size } w \text{ tree surviving in } H^{(3)}] \le \frac{1}{n}$ $\le \frac{n}{2^w}$

$\Rightarrow$ $\Pr[\exists$ size $w \cdot d^3$ component surviving in $G] \le \frac{1}{n}$

So,

unlikely to have any surviving component of size $\Omega(d^3 \log n)$.

∎