Sp 2014

Lecture 6

Random Walks!

Markov chains + random walks on graphs

Stationary Distributions

Hitting, Cover, Commute Time

Random walks

Markov chains:

Ω = set of "states" (or nodes) (here always FINITE)

$X_0 \dots X_t \in \Omega$ sequence of visited states

Markovian property:

$$\Pr[X_{t+1} = y \mid X_0 = x_0, X_1 = x_1, X_2 = x_2, \dots, X_t = x_t] = \Pr[X_{t+1} = y \mid X_t = x_t]$$

Next step depends only on where you are. Not how you got there.

Wlog, assume transitions independent of time:

i.e. $P(x,y) = \Pr[X_{t+1} = y \mid X_t = x]$

so can use "transition matrix" to represent it

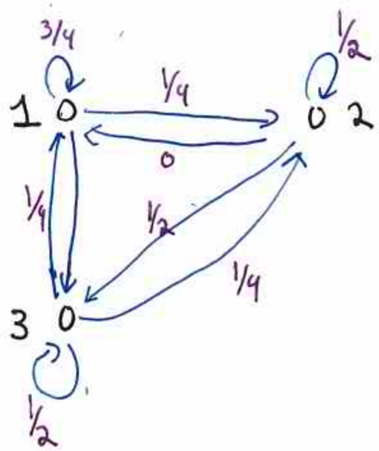
Important special case:

transitions uniform on subset corresponding to neighbors of node

def. random walk on $G = (V, E)$
is a sequence $s_0\, s_1 \dots$ of nodes
where s_0 is a start node.
At each step i, s_{i+1} picked uniformly from $\underbrace{N(s_i)}_{\text{outedges}}$

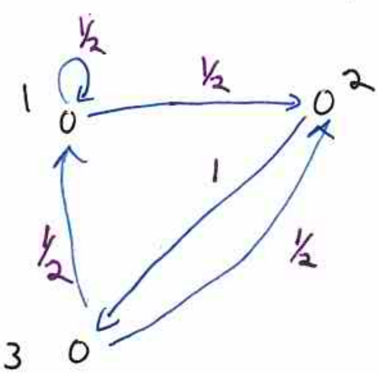
examples

Markov chain



$$p: \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 3/4 & 1/4 & \\ 2 & 0 & 1/2 & 1/2 \\ 3 & 1/4 & 1/4 & 1/2 \end{array}$$

random walk on digraph



$$p: \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1/2 & 1/2 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1/2 & 1/2 & 0 \end{array}$$

$d(i) =$ # outedges of node i

$$p(i,j) = \begin{cases} \frac{1}{d(i)} & \text{if } (i,j) \in E \\ 0 & \text{o.w.} \end{cases}$$

$$\forall i \quad \sum_j p(i,j) = 1$$

Distributions after t steps

Transition probabilities for ~~t~~ steps: $P^t(x,y) = \begin{cases} P(x,y) & t=1 \\ \sum_z P(x,z)P^{t-1}(z,y) & t>1 \end{cases}$ matrix multiplication $P^t = \underbrace{P \times P \times \dots \times P}_{t \text{ times}}$

Initial distribution: $\pi^0 = (\pi^0_1 \dots \pi^0_n)$ where $\pi^0_i = \Pr[\text{start at node } i]$

distribution after one step:

$$\pi^1 = \pi^0 \cdot P = \left(\sum_z P(z,1)\cdot\pi(z), \sum_z P(z,2)\pi(z), \dots \right)$$

⋮

t-step distribution: $\pi^t = \pi^0 \cdot P^t$

Finite Markov Chain Properties

stochastic matrix: rows of P sum to 1 — all M.C.'s have this property

doubly stochastic matrix: rows & columns sum to 1 — not even all interesting M.C.'s satisfy this

e.g. random walk on undirected graph
or digraph in which
indegree = outdegree = const for all nodes

irreducible: ("strongly connected")

$$\forall x,y \ \exists t = t(x,y) \ \text{s.t.} \ P^t(x,y) > 0$$

↕ change of quantifier order

ergodic: $\exists t_0 \ \text{s.t.} \ \forall t > t_0 \ \forall x,y \quad P^t(x,y) > 0$ ← stronger than irreducible

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gcd of "possible" cycle length =1

Aperiodic:

$\forall x \quad \gcd \{ t : P^t(x,x) > 0 \} = 1$ — not bipartite, k-partite...

<u>Thm</u> Ergodic $\iff$ Irreducible + Aperiodic

Stationary Distributions

does it depend on π_0?

stationary distribution π — so $\pi^t = \pi^{t-1}$

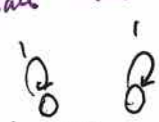
$$\forall y \quad \pi(y) = \sum_x \pi(x) P(x,y)$$

Will consider P s.t. π is unique & exists — i.e. doesn't depend on π_0

if periodic: could have no stat. dist. or several

if $\pi_0 = (0,1)$ then $\pi_{2i} = (0,1)$, $\pi_{2i+1} = (1,0)$

if reducible: could have lots of stat. dist.



Some stat dist's: $(\frac{1}{2}, \frac{1}{2})$ $(0,1)$ $(1,0)$...

<u>Important Thm</u> every ergodic M.C. has unique stationary distribution

Stationary dist. of undirected graph:

$$\pi = \left(\frac{\deg(x_1)}{2|E|}, \frac{\deg(x_2)}{2|E|}, \ldots \right)$$

- so d-regular graphs have π = uniform
  (also indegree = outdegree = d digraphs
  & $\underbrace{\text{doubly stochastic}}_{\text{this implies the others!}}$ P M.C.'s)
- not true in general for digraphs
- bipartite, periodic graphs may have other stat. dists.

Hitting times

h_{ii} = E [time starting at i to return to i]

$= \dfrac{1}{\pi_i}$ ← Very useful theorem!

h_{ij} = E [time starting at i to reach j]

Cover time of undirected graph

$\mathcal{C}_u(G)$ = E [# steps to reach all nodes in G on walk starting from u]

$$\mathcal{C}(G) = \max_u \mathcal{C}_u(G)$$

Cover time Examples:

- $\mathcal{C}(K_n^*)$ where K_n^* = complete graph with self-loops at each node } so aperiodic

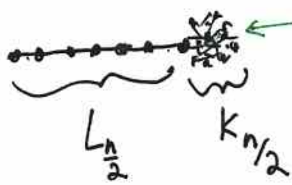
  $= \Theta(n \ln n)$ by coupon collector argument

- $\mathcal{C}(L_n)$ where L_n = n node line

  $= \Theta(n^2)$

- $\mathcal{C}$ (lollipop)

  $= \Theta(n^3)$



← worst place to start

Thm $\mathcal{C}(G) \leq 8m(n-1)$

Proof

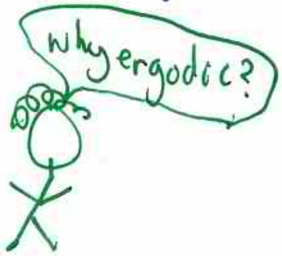
First - transform G into G' (see example on pg 8)

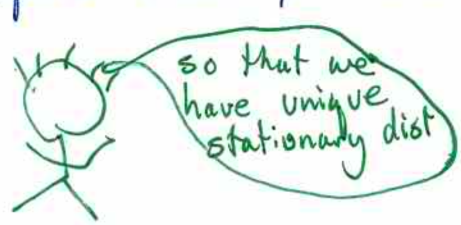
to make G aperiodic, add d_u self loops to each u

(i.e. take self-loop with prob $1/2$)

Claim: $\mathcal{C}(G') = 2\,\mathcal{C}(G)$

why? can transform paths in G' by removing self-loops, expected # self-loops = $1/2$ (length of path)





Next, commute times + a lemma:

def. C_{ij} = E[#steps for r.w. starting at i to hit j + return to i] "commute time"

Claim. $C_{ij} = h_{ij} + h_{ji}$ (linearity of expectation)

Lemma $\forall (u,v) \in E \quad C_{uv} \le O(m)$

Pf of lemma

Key idea:
if traverse (u,v) twice

$(u,v) \;\dots\; (u,v)$

have performed commute from $u \rightsquigarrow v \rightsquigarrow u$

Plan: show E[time between visits to (u,v)] is $O(m)$

$\Longrightarrow C_{uv}$ is $O(m)$

Given $G' = (V,E)$ (G with added self loops)

Construct (directed) G'' representing walks on directed edges of G'

line graph
$E \to V''$ new nodes $V_{(u,v)}$ are edges (u,v) in G'

$\underbrace{(u,v)(v,w)}_{\text{consecutive edges}} \to E''$ new edges are length 2 paths in G'

visit edge in G' twice $\Leftrightarrow$ visit node in G'' twice example

G

$1 \to 2 \to 1$

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0 & 1 \\ 2 & 1 & 0 \end{array}$$

$\Rightarrow$

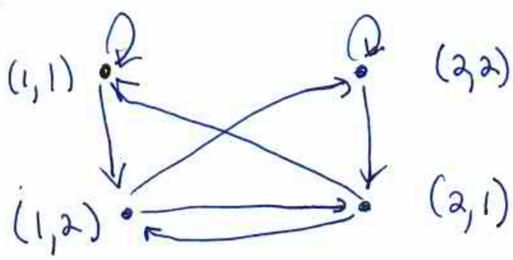
G'

$1 \to 1 \to 2 \to 2 \to 2 \to 1$

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 1/2 & 1/2 \\ 2 & 1/2 & 1/2 \end{array}$$

$\Downarrow$

G''



	(1,1)	(1,2)	(2,2)	(2,1)
(1,1)	1/2	1/2	0	0
(1,2)	0	0	1/2	1/2
(2,2)	0	0	1/2	1/2
(2,1)	1/2	1/2	0	0

(more complicated example)

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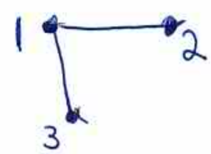
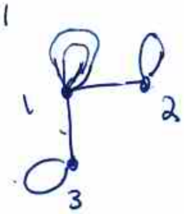
example

G



$\Rightarrow$

G'



$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1/2 & 1/4 & 1/4 \\ 2 & 1/2 & 1/2 & 0 \\ 3 & 1/2 & 0 & 1/2 \end{array}$$

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1/2 & 1/2 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{array}$$

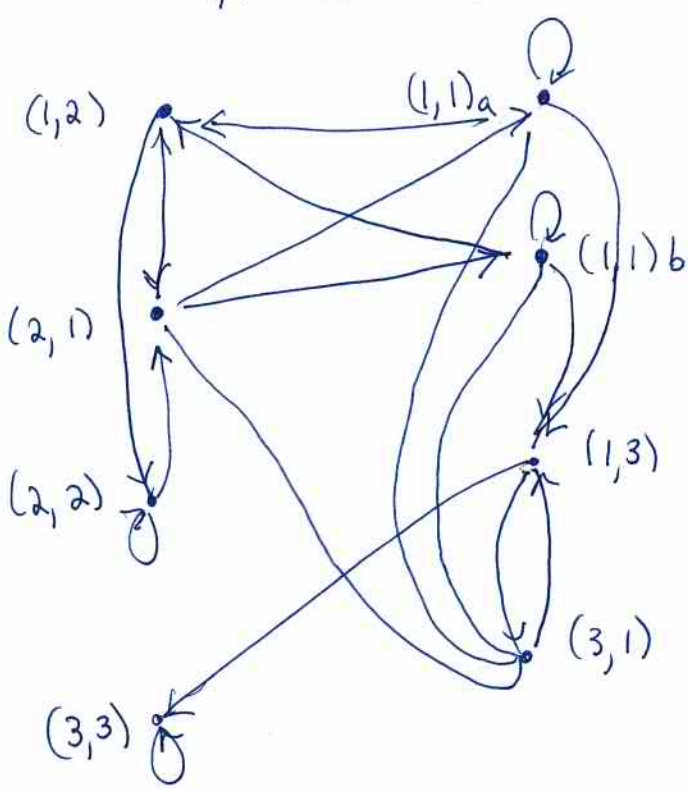
$1 \to 2 \to 1$

$1 \to 1 \to 2 \to 1$

$\Downarrow$

$(1,1) \to (1,2) \to (2,1)$

G''



Note: G'' is doubly stochastic:

$$Q_{(u,v)(v,w)} = P_{vw} = \frac{1}{d(v)} \quad \text{if} \quad (u,v)\,(v,w) \in E$$

$$\forall\, (v,w) \in \underset{\substack{\parallel \\ V''}}{E} \qquad \underbrace{\sum_{\substack{(u,v) \text{ s.t.} \\ (u,v)(v,w) \in E'}}}_{\text{column sum}} Q_{(u,v)(v,w)} = \sum_{(u,v)\in E} \frac{1}{d(v)} = 1$$

$\therefore\ \Pi$ of G'' is uniform

we need that walk on G'' is ergodic. Irreducible follows from G' irreducible. Aperiodic comes from self-loops.

$$\text{so} \quad \Pi_{\underset{\substack{\uparrow \\ \text{edge in } G'}}{u}} = \frac{1}{|V''|} = \frac{1}{4\underset{\substack{\uparrow \\ \#\text{ edges in } G}}{m}}$$

$$h_{\underset{\substack{\uparrow \\ \text{edge in } G'}}{uu}} = \frac{1}{\Pi_u} = 4m \qquad \text{for all nodes } \underset{\substack{\parallel \\ (a,b)}}{u} \text{ in } G''$$

(a,b) in G'

$\blacksquare$
(proof of lemma)

so start at u, walk $\leq 4m$ steps until hit (u,v) then another $\leq 4m$ until hit (u,v) again

Note: valid only for $(u,v) \in E$

Wrapping it up:

Lemma $C(G) = O(nm) = O(n^3)$

Pf.

start vertex v_0

$T \leftarrow$ spanning tree rooted at v_0

$\#$ edges in $T = n-1$

$v_0 \; v_1 \cdots v_{2n-2}$ is depth 1st traversal
$\qquad\qquad\; \overset{\shortparallel}{v_0}$

st. each edge appears twice, once in each direction (a,b) + (b,a)

$$C(G) \le \sum_{j=0}^{2n-3} h_{v_j v_{j+1}}$$

$$= \sum_{(u,v) \in T} C_{uv} \qquad \text{since } C_{uv} = h_{uv} + h_{vu}$$

$$= O\Big(\sum_{(u,v)\in T} m\Big)$$

$$= O(nm) \qquad \blacksquare$$