6.895 Randomness and Computation	March 22, 2006
Lecture 13	
Lecturer: Ronitt Rubinfeld	*Scribe: Punyashloka Biswal*

1 Conductance

For a graph to have a small mixing time, we would like a random walk that starts within some small subset of nodes to quickly have a non-zero probability of being anywhere on the graph. To capture this idea, we define the notion of *conductance* as follows:

Definition 1 (Conductance, first attempt) *Let $G = (V, E)$ be an undirected graph and $S \subset V$ be a set of nodes. Then the conductance $\Phi(S, \bar{S})$ is defined as*

$$\Phi(S, \bar{S}) = \frac{|E|_{S,\bar{S}}}{|E|_S},$$

where

$$\begin{aligned} \bar{S} &= V - S, \\ E_S &= \{(u, v) \in E \mid u, v \in S\} \text{ and} \\ E_{S,\bar{S}} &= \{(u, v) \in E \mid u \in S, v \in \bar{S}\}. \end{aligned}$$

The conductance of the graph Φ_G is defined as

$$\Phi_G = \min_{|S| \le |V|/2} \Phi(S, \bar{S}).$$

To see why a graph with large conductance should have a small mixing time, let S be the set of 'overweight' nodes v such that $\pi_v^t > \tilde{\pi}_v$. Since G has a large conductance, there are many ways for a random walker on S to cross over to $\bar{S}$ and reduce the probability gap. In the extreme case where $\pi_v^t = 1/|S|$ for $v \in S$ and zero otherwise, then the probability of crossing the cut is precisely the conductance $\Phi(S, \bar{S})$.

The definition of Φ_G restricts the mimimum to subsets S of at most $|V|/2$ vertices to make sure that our results are not skewed by overly large sets. For example, consider $S = V - \{v\}$ when G is d-regular: clearly, $\Phi(S, \{v\}) = d/[(n-1)d] = 1/(n-1)$, which is very small regardless of the large-scale properties of the graph. To get around this problem, we only compute conductances between subsets that form a constant fraction of the entire graph (the choice of the value $1/2$ is arbitrary).

In order to avoid this unnatural restriction, as well as to make the conductance symmetric with respect to cuts (so that $\Phi(S, \bar{S}) = \Phi(\bar{S}, S)$), we shall henceforth use a somewhat different definition:

Definition 1′ (Conductance) *Let $G = (V, E)$ and S be defined as in definition 1. Then the conductance of the cut $(S, \bar{S})$ is defined as*

$$\Phi_S = \Phi_{\bar{S}} = \frac{|E_{S,\bar{S}}||E|}{|E_S||E_{\bar{S}}|}$$

and the graph conductance Φ_G is defined as the minimum conductance over all cuts.

Without loss of generality, suppose $|E_S| \le |E_{\bar{S}}|$. But $E = E_S \cup E_{\bar{S}}$, so that $|E|/|E_{\bar{S}}| \le 2$. This implies that the new definition differs from the old one by a factor of at most 2.

Definition 2 ($\mathcal{L}_2$-Distance) *The $\mathcal{L}_2$-distance between two distributions D_1 and D_2 over a discrete set X is denoted by $\|D_1 - D_2\|_2$ and is defined as*

$$\|D_1 - D_2\|_2 = \sqrt{\sum_{x \in X} \big(D_1(x) - D_2(x)\big)^2}$$

We are usually interested in the $\mathcal{L}_1$-distance between probability distributions, and the following lemma relates the two notions of distance:

Lemma 3 *Let D_1 and D_2 be two distributions. Then*

$$\|D_1 - D_2\|_2 \leq \|D_1 - D_2\|_1 \leq \sqrt{n}\|D_1 - D_2\|_2$$

Proof Write $D = D_1 - D_2$ and $D(x) = D_1(x) - D_2(x)$. Then, on one hand,

$$\|D\|_1^2 = \left(\sum |D(x)|\right)^2 = \sum D(x)^2 + \sum_{x \neq y} |D(x)||D(y)| \geq \sum D(x)^2 = \|D\|_2^2.$$

On the other hand, if we apply Chebychev's sum inequality to the numbers $|D(x_1)|, |D(x_2)|, \ldots, |D(x_n)|$ and $1, 1, \ldots, 1$, then we get

$$\left(\sum_{x \in X} |D(x)|\right)^2 \leq n \sum_{x \in X} |D(x)|^2,$$

or $\|D\|_1^2 \leq n\|D\|_2^2$. Taking the square roots of these two inequalities, we have the result. ■

The following theorem (which we shall prove in a subsequent lecture) gives a precise relationship between the conductance of a graph and the mixing time:

Theorem 4 *Let P be the transition matrix corresponding to a random walk on a graph G, and define $d(t) = \|P^t\pi_0 - \tilde{\pi}\|_2^2$ to be the square of the L_2-distance between the distribution after t steps and the stationary distribution. Then*

$$d(t) \leq \left[1 - \frac{\Phi_G^2}{4}\right]^t d(0)$$

Notice that $d(0) \leq 2$ for all starting distributions π_0, because

$$\|\pi_0 - \tilde{\pi}\|_2 \leq \|\pi_0\|_2 + \|\tilde{\pi}\|_2 \leq \|\pi_0\|_1 + \|\tilde{\pi}\|_1 = 2.$$

Therefore, if we set $t = (4/\Phi_G^2)\ln(2n/\varepsilon^2)$, then

$$d(t) \leq \left[1 - \frac{\Phi_G^2}{4}\right]^{\frac{4}{\Phi_G^2}\ln\frac{2n}{\varepsilon^2}} \cdot 2 \leq \frac{\varepsilon^2}{n}$$

by theorem 4. We can now apply lemma 3 to translate this into an L_1 bound:

$$\left\|P^t\pi_0 - \tilde{\pi}\right\|_1 \leq \sqrt{n}\left\|P^t\pi_0 - \tilde{\pi}_1\right\|_2 = \sqrt{nd(t)} \leq \varepsilon.$$

This formalizes our earlier intuition that a graph with a large conductance mixes fast. More specifically, it suffices to show that $\Phi_G = \Omega(1/\log n)$ to prove rapid mixing. In some cases, we can even show a *constant* lower bound on the conductance!

We shall be particularly interested in graphs that are d-regular for some d. In this case, the conductance is given by

$$\Phi_G = \min_S \frac{|E_{S,\bar{S}}||E|}{|E_S||E_{\bar{S}}|} = \min_S \frac{|E_{S,\bar{S}}|d|V|}{d|S|d|\bar{S}|} = \frac{1}{d}\left(\min_S \frac{|E_{S,\bar{S}}||V|}{|S||\bar{S}|}\right).$$

The parenthetized term above has a special name: it is the *edge magnification* μ:

$$\mu = \min_S \frac{|E_{S,\bar{S}}||V|}{|S||\bar{S}|},$$

and for d-regular graphs, $\Phi_G = \mu/d$.

One important technique for lower-bounding the conductance of a graph is the method of canonical paths, which we have already used for the hypercube. The idea is to carefully choose a set of paths between every pair of nodes, such that no edge in the graph has too many paths going through it:

Definition 5 (Congestion) *Let $P = \{p_{uv}\}$ be a set of canonical paths for a graph $G = (V, E)$, where p_{uv} connects vertex u to vertex v. Then the congestion of an edge $e \in E$ is defined as the number of paths $p \in P$ that use e. Also, the congestion of G is defined as the maximum congestion over all edges e.*

The congestion of a graph can be as large as $O(n^2)$—consider, for example, the line on n nodes—but for many graphs, it is possible to find a set of canonical paths that makes the congestion small. For a graph of low conductance, however, there are bottleneck edges which must be congested by *any* chosen set of paths.

Claim 6 *If G has congestion αn with respect to some set of canonical paths, then $\mu \geq 1/\alpha$.*

Proof Fix a cut $(S, \bar{S})$ of G. Then the number of canonical paths p_{uv} connecting $u \in S$ to $v \in \bar{S}$ is $|S||\bar{S}|$. Each of these paths has to use at least one edge e in the cut, i.e., $e \in E_{S,\bar{S}}$. By the definition of congestion, we have

$$\begin{aligned}
\#\text{ of paths crossing cut} &\leq \sum_{e \in E_{S,\bar{S}}} (\#\text{ of paths crossing } e) \\
&\leq |E_{S,\bar{S}}| \max_{E_{S,\bar{S}}} (\#\text{ of paths crossing } e) \\
|S||\bar{S}| &\leq |E_{S,\bar{S}}|\alpha n \\
\frac{|E_{S,\bar{S}}|n}{|S||\bar{S}|} &\geq \frac{1}{\alpha}
\end{aligned}$$

for all cuts $(S, \bar{S})$. The edge expansion μ is the minimum value of the left hand side of the above inequality, so $\mu \geq 1/\alpha$. ■

Recall that in lecture 7 (weakly learning monotone functions), we studied the conductance of the hypercube on $n = 2^N$ nodes using canonical paths. We chose paths which had the property that an edge on a path, along with N additional bits (or a *complementary point*), completely determined the start and end node (and therefore the path). This property, allowed us to argue that no more than n distinct paths could pass through a given edge, bounding the congestion and hence the conductance. We will do something similar for the problem of uniformly generating graph matchings, which we shall address next.

2 Uniformly Generating Matchings

Given a bipartite graph $G = (V, E)$ where $m = |E|$, we wish to generate a matching of the vertices of the graph uniformly at random.[1] We do this by constructing a Markov chain with states corresponding to matchings and in which transitions correspond to small local changes in the matching. Given an initial matching (state) M, the possible transitions are defined as follows:

Pick an edge $e \in_R E$
if $e \in M$,
  then set $M \leftarrow M - \{e\}$
else if $M \cup \{e\}$ is a matching
  then set $M \leftarrow M \cup \{e\}$
else
  stay put

The resulting Markov chain $\mathcal{M} = (\mathcal{S}, \mathcal{T})$ has the following properties:

- It is *undirected*, because every transition is reversible.
- It is *connected*: to get from matching M_1 to M_2: Drop all the edges in M_1 to get to the empty matching, and then build up M_2 one edge at a time. In fact, this shows that the diameter of the chain is at most $2|M| \leq 2|V|/2 = |V|$, where M is a maximal matching.
- It is *non-bipartite*, because it has at least one self-loop (for example, consider starting from any maximal matching and picking an edge not in the matching).
- It is *regular* with degree m, because for any initial matching, we can consider any of the m edges of G to add or remove.

In order to define the canonical paths on this Markov chain, we note that the symmetric difference $M_1 \oplus M_2$ of two matchings consists of a set of alternating paths and cycles. We fix an arbitrary ordering on the edges of G, a start edge for every possible path or cycle, and a traversal direction for every cycle.

To convert M_1 into M_2, we consider the edges in $M_1 \oplus M_2$ in the order defined above. When we encounter an edge e, we process the entire alternating path or cycle that contains it (as shown below). We keep doing this until there are no more paths or cycles to process.

- To process a path $e_1 e_2 \ldots e_k$, we have to delete an edge before we can add a new one. Assume e_1 and e_k both must be added. If not, we can just delete them before running the algorithm. So k is odd. The algorithm is:

  $i \leftarrow 1$
  while $i \neq k$ **do**
    Delete e_{i+1}
    Insert e_i
    $i \leftarrow i + 2$
  Insert e_k

- To process a cycle $e_1 e_2 \ldots e_k e_1$, we need to be careful, because we must delete *two* edges in the cycle before any insertions are possible. Assume e_1 must be deleted. Note that k must be even. The algorithm runs as follows:

[1] It is possible to generate maximal and/or perfect matchings, but here we address the simpler problem of generating arbitrary matchings.

Delete e_1
$i \leftarrow 2$
while $i \neq k$ **do**
  Delete e_{i+1}
  Insert e_i
  $i \leftarrow i + 2$
Insert e_1

Given a transition $e \in \mathcal{T}$, we need to find a way to bound its congestion. We shall do so by answering the question: "what additional information do we need to reconstruct the endpoints of the path?" For the hypercube, we found this bound in terms of a number of bits, but in this case, we don't even know how large $\mathcal{S}$ is. Luckily, however, claim 6, which bounds the conductance, requires the value of the congestion to be specified as a multiple of the chain size. Therefore, we shall specify the aditional information in the form of *another* matching (the complementary point) and a small number of additional bits.

Claim 7 *Fix a transition $M_a \to M_b$. We can reconstruct the starting and ending states M_1 and M_2 of the canonical path if we specify the additional information $\bar{M} = (M_1 \oplus M_2) - M_a$.*

Proof Using the ordering on edges, we can decide which edges in M_a have not yet been corrected. These edges must match M_1. The remaining edges of M_1 are given by the corrections contained in $M_a \oplus \bar{M}$. Similarly, we can reconstruct M_2 as well. ■

Unfortunately, we are not quite done, because $\bar{M}$ might not be a matching, so that it is unsuitable as a complementary point. However, it can be shown that we can always remove at most two edges from $\bar{M}$ to make it into a matching. Therefore, it suffices to specify the resulting matching, along with one of m^2 possibilities for the two edges. This means that the edge congestion is at most $m^2|S|$. By claim 6, $\mu \geq 1/m^2$. We have already noted that $\mathcal{M}$ is m-regular, so that

$$\Phi_G = \frac{\mu}{m} = \frac{1/m^2}{m} = \frac{1}{m^3}.$$

The number of matchings is bounded by the number of subsets of the edge set, 2^m. Using this, we can set

$$t = \frac{4}{\Phi_G^2} \ln \frac{2|S|}{\varepsilon^2} \leq 4m^6 \ln \frac{2^{m+1}}{\varepsilon^2} = O(m^7 \ln(1/\varepsilon))$$

to get within ε of the uniform distribution. Therefore, the Markov chain mixes rapidly, or in polynomial time.