Lecture 3

Sublinear time approximation of average degree

Warning: this is a different algorithm than described in notes for lecture 2.
hopefully simpler!!

Estimating Average Degree

Given $G = (V, E)$

$\varepsilon \in (0,1)$ approximation parameter

$\delta \in (0,1)$ confidence $\leftarrow$ lets assume $\delta = 1/4$

Output $\tilde{d}$ s.t. $\Pr[|\tilde{d} - \bar{d}| \leq \varepsilon \bar{d}] \geq 1 - \delta$

where $\bar{d} = \frac{m}{n}$ (average degree)

Assumption: (1) $\bar{d} \geq 1$

(2) given access to

- "degree queries": given x outputs $\deg(x)$
- "neighbor queries": given (v, j) output j^{th} nbr of v

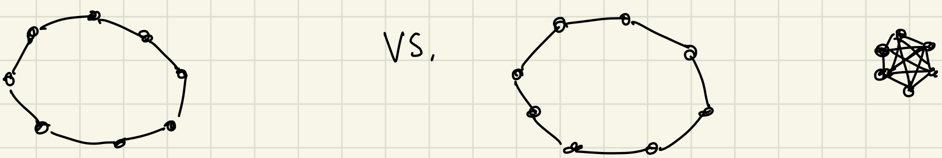
Last time:

Question: naive sampling needs $\Omega(n)$ samples??

Lower bound:

Distinguish

n-cycle $\bar{d}=2$ vs. $n-c\sqrt{n}$-cycle + $c\sqrt{n}$-clique $\bar{d} \approx 2+c^2$



need $\Omega(\sqrt{n})$ queries to distinguish?

ignore notes for algorithm last time

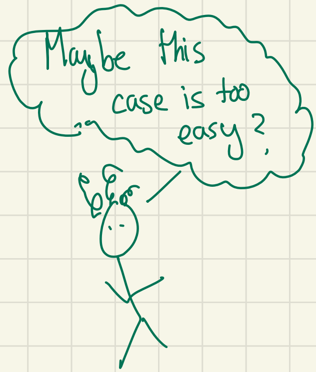
today will do simpler algorithm

Today:

Warm up: regular graphs

Assume each node has degree d

Algorithm: output d



Better warmup: almost regular graphs

Assume each node has degree in $[\Delta, 10\Delta]$

Algorithm:

$K \leftarrow \frac{50}{\varepsilon^2} \ln(2/\delta)$

For $i \leftarrow 1$ to K do

- pick $v_i \in_u V$
- $X_i \leftarrow \deg(v_i)$

Output $\tilde{d} \leftarrow \frac{1}{K} \sum_{i=1}^{K} X_i$

notation:
$x \in_u D$
means pick x uniformly from set D

Runtime: $O\left(\frac{1}{\varepsilon^2} \ln \frac{1}{\delta}\right)$

Behavior:

Claim $E[\tilde{d}] = \bar{d}$

Pf

$$E[\tilde{d}] \underset{\text{lin of exp}}{=} \frac{1}{k} \sum_{i=1}^{k} E[X_i] \underset{\text{iid}}{=} E[X_1]$$

$$= \sum_{v \in V} \frac{1}{n} \deg(v) = \frac{\sum \deg(v)}{n} = \bar{d} \quad \blacksquare$$

Claim $\Pr[\,|\bar{d} - \tilde{d}| \leq \varepsilon \bar{d}\,] \geq 1-\delta$

Pf

Will use following version of Chernoff Bnd:

Thm let $Y_1 \dots Y_k$ be independent random variables s.t. $Y_i \in [0,1]$ & $Y = \sum_{i=1}^{k} Y_i$. For $b \geq 1$

$$\Pr[\,|Y - E[Y]| > b\,] \leq 2 \cdot \exp(-2b^2/k)$$

so can't use Chernoff

Note: X_i's are $\underline{\text{not}}$ in $[0,1]$ but are in $[\Delta, 10\Delta]$

let $Z_i \leftarrow \frac{X_i}{10\Delta}$ then $Z_i \in [0,1]$

$$Z \leftarrow \sum_{i=1}^{k} Z_i \qquad \tilde{d} = \frac{10\Delta}{k} \cdot Z$$

$$E[Z] = \frac{k}{10\Delta} \cdot E[\tilde{d}] = \frac{k\bar{d}}{10\Delta}$$

$$|\tilde{d} - \underset{\uparrow\, E[\tilde{d}]}{\bar{d}}| \geq \varepsilon \bar{d} \iff \left|\frac{10\Delta}{k} Z - \frac{10\Delta}{k} E[Z]\right| \geq \varepsilon \bar{d}$$

$$\iff |Z - E[Z]| \geq \frac{k}{10 \cdot \Delta} \cdot \varepsilon \bar{d}$$

Use Chernoff on Z's

with $b = \frac{k}{10\Delta} \varepsilon \bar{d}$

$$\Pr\left[|Z - E[Z]| \geq \frac{k}{10\Delta} \varepsilon \bar{d}\right] \leq 2 \cdot e^{-\left(\frac{2k^2 \varepsilon^2 \bar{d}^2}{100\Delta^2 \cdot k}\right)}$$

$$= 2e^{-\frac{1}{50} \cdot \frac{k \varepsilon^2 \bar{d}^2}{\Delta^2}}$$

$\bar{d} \geq \Delta$ by assumption on all degrees $\geq \Delta$

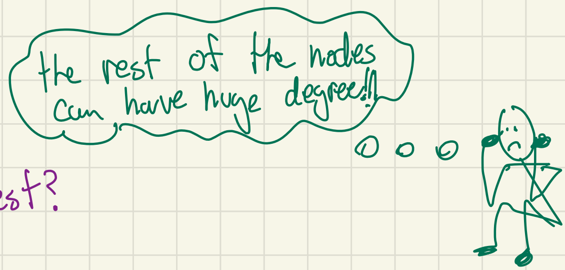
$$\leq 2e^{-\frac{k\varepsilon^2}{50}}$$

$$= 2e^{-\frac{(\cancel{50}/\cancel{\varepsilon^2})(\ln 2/\delta) \cdot \cancel{\varepsilon^2}}{\cancel{50}}} = \delta \qquad \blacksquare$$

General Case:

by Markov's $\neq$, $\leq \frac{1}{c}$ nodes have degree $\geq c \cdot \bar{d}$. Can we use that?

- so most nodes satisfy warmup case!
- what about the rest?



define total order "$\prec$" on nodes:

assume distinct ID's

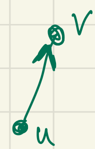
def. $u \prec v$ if

- $\deg(u) < \deg(v)$

or

- $\deg(u) = \deg(v)$ & $ID(u) < ID(v)$

$\deg^+(u)$ = # nbrs of u s.t. $u \prec v$



Orienting edges from small to large, $\deg^+(u)$ counts "out-edges"

Observation $\sum_{u \in V} \deg^+(u) = m = \frac{n}{2} \cdot \overline{d}$

(since each edge only counted once instead of twice as in $\sum_u \deg(u)$)

idea estimate $\operatorname{average}_u \left(\deg^+(u) \right)$

problem? we can query $\deg(u)$ not $\deg^+(u)$

benefit:

Lemma $\forall v \in V \quad \deg^+(v) \leq \sqrt{m}$

Proof

define $H \subseteq V$ to be $\sqrt{2m}$ nodes with highest rank (degree) w.r.t. $\prec$

$\forall v \in H, \deg^+(v) \leq \sqrt{2m}$ since edges "leaving" v go to bigger nodes

(which must also be in H)

$\forall v \in V \setminus H$, $\deg^+(v) \le \deg(v) \le \sqrt{2m}$:

Why? if not, $\deg(v) > \sqrt{2m}$ ← assume for contradiction

but all w in H have

$$\deg(w) \ge \deg(v) > \sqrt{2m}$$

so total degree

$$> \underbrace{|H| \cdot \sqrt{2m}}_{\text{contribution from } H} + \underbrace{\text{something positive}}_{\text{contribution from } V \setminus H}$$

$$> \sqrt{2m} \cdot \sqrt{2m} = 2 \cdot m$$

but sum of degrees $= 2 \cdot m$

$\underbrace{\rightarrow \leftarrow}_{\text{symbol for "contradiction"}}$ ■

Algorithm:

$K \leftarrow \frac{16}{\varepsilon^2} \sqrt{n}$

for $i = 1$ to K

pick $v_i \in_r V$ (1)

pick $u_i \in_r N(v_i)$ (2)

if $u_i \prec v_i$ then $X_i \leftarrow 2\deg(v_i)$

else $X_i \leftarrow 0$

return $\tilde{d} = \frac{1}{K} \sum_{i=1}^{K} X_i$

Question to think about: why the "2"?

Claim $E[X_i] = \bar{d}$

Pf

$$E[X_i] = \sum_{v \in V} \Pr[v \text{ chosen in } (1)] \cdot E[X_i \mid v \text{ chosen in } (1)]$$

$$= \sum_{v \in V} \frac{1}{n} \cdot E[X_i \mid v \text{ chosen in } (1)]$$

$$= \frac{1}{n} \sum_{v \in V} \sum_{u \in N(v)} \Pr[u \text{ chosen in } (2) \mid v \text{ chosen in } (1)]$$
$$\times \underbrace{E[X_i \mid u \text{ chosen in } (2) \text{ \& } v \text{ chosen in } (1)]}_{\text{if } v \prec u \text{ then } 2\deg(v) \text{ else } 0}$$

$$= \frac{1}{n} \cdot \sum_{v \in V} \sum_{\substack{u \in N(v) \\ \& \, v \prec u}} \frac{1}{\deg(v)} \cdot 2 \cdot \deg(v)$$

$$= \frac{2}{n} \cdot \sum_{v \in V} \deg^+(v) = \frac{2m}{n} = \bar{d}$$

∎

But how many samples do we need to assure that we are close to expectation? Here is where we use graph properties!

<u>Claim</u> $Var[X_i] \leq 4\sqrt{2m}\,\bar{d}$

<u>Pf</u> $Var[X_i] = E[X_i^2] - E[X_i]^2 \leq E[X_i^2]$ as above

$$= \frac{1}{n} \sum_{v \in V} \sum_{\substack{u \in N(v) \\ v \prec u}} \frac{1}{deg(v)} \underbrace{(2\,deg(v))^2}_{X_i^2}$$

$$= \frac{4}{n} \sum_{v \in V} \underbrace{deg^+(v)}_{\leq \sqrt{2m}} \cdot deg(v)$$ ← key insight

$$\leq \frac{4}{n} \cdot \sqrt{2m} \sum_{v \in V} deg(v)$$

$$\leq 4 \cdot \sqrt{2m} \cdot \bar{d}$$

∎

2 useful facts about variance:

- <u>Lemma</u> let $Y = \frac{1}{k}\sum_{i=1}^{k} X_i$ where X_i's are $\underline{\text{iid}}$ — important but pairwise independence is good enough

  so can reduce variance by sampling & averaging more! → then $Var[Y] = \frac{1}{k} Var[X]$

- Chebyshev's ≠: $Pr[\,|X - E[X]| \geq b\,] \leq \frac{Var[X]}{b^2}$

Lemma $\Pr\left[\,|\tilde{d}-\bar{d}| \le \varepsilon \bar{d}\,\right] \ge 3/4$

Pf

$E[\tilde{d}] = \bar{d}$ by lin of expectation

$$\mathrm{Var}[\tilde{d}] \le \frac{4\cdot\sqrt{2m}}{k}\cdot \bar{d}$$

since $\bar{d} = E[\tilde{d}]$

$$\Pr\left[\,|\tilde{d}-\bar{d}| \ge \varepsilon\bar{d}\,\right] = \Pr\left[\,|\tilde{d} - E[\tilde{d}]| \ge \varepsilon \bar{d}\,\right]$$

$$\le \frac{\mathrm{Var}[\tilde{d}]}{(\varepsilon\bar{d})^2}$$

$$\le \frac{\frac{4\sqrt{2m}}{k}\cdot\bar{d}}{\varepsilon^2\bar{d}^2} = \frac{4\sqrt{2m}}{\varepsilon^2\cdot\underbrace{\bar{d}}_{=\frac{2m}{n}}\cdot k}$$

$$= \frac{4\sqrt{2m}\cdot n}{\varepsilon^2\cdot 2m\cdot k} = \frac{4n}{\varepsilon^2\cdot\sqrt{2m}\cdot k}$$

pick $k = \frac{16}{\varepsilon^2}\sqrt{n}$

$$= \frac{\sqrt{n}}{4\cdot\sqrt{2m}}$$

$$\le \frac{1}{4}$$

since $\sqrt{\frac{n}{2m}} = \sqrt{\frac{1}{\bar{d}}} \le 1$ since we assumed $\bar{d} \ge 1$

How do we improve probability of success?

see HW 0!