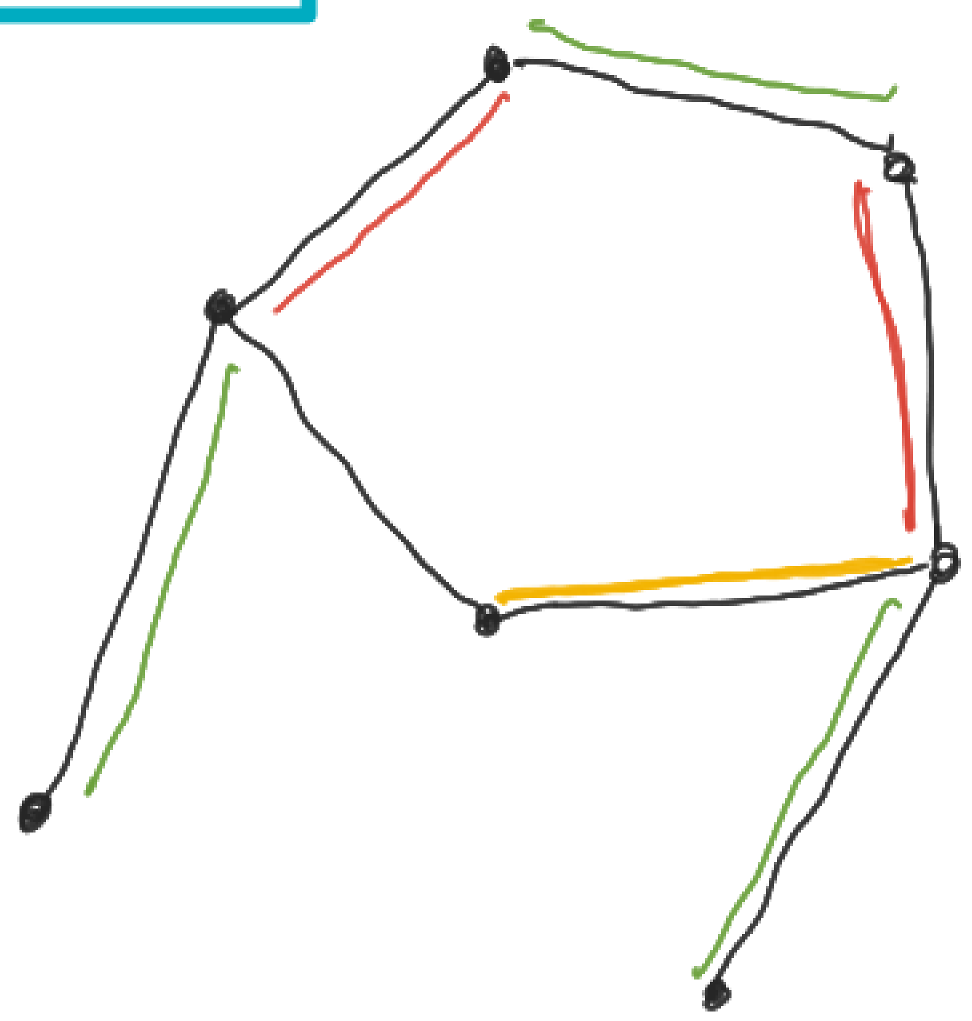
Matchings

Definition

$G = (V, E)$

MATCHING = A SET OF DISJOINT EDGES



MATCHING

MAXIMAL MATCHING

MAXIMUM MATCHING

|MAXIMUM MATCHING| $\leq$ 2 · |MAXIMAL MATCHING|

Greedy algorithm

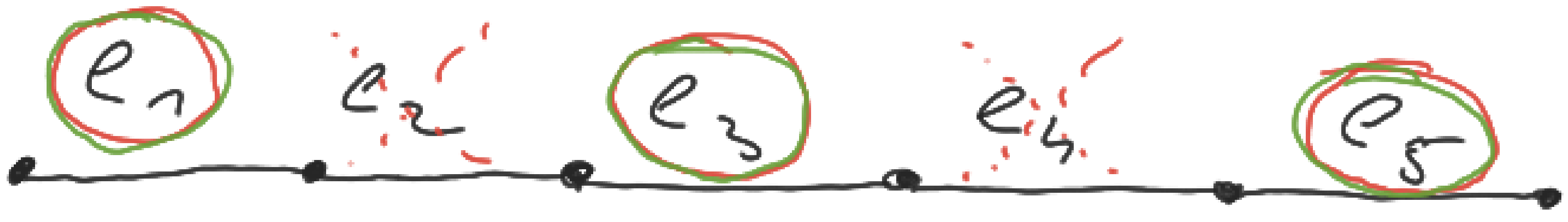
$G = (V, E)$

(1) $M = \emptyset$

(2) GO OVER E IN AN ARB. ORDER

(3) IF THE CURRENT EDGE e DOES NOT INTERSECT M, THEN

(3.1) $M = M + e$



GREEDY ALG. IS VERY SEQUENTIAL.

Can we design "less" sequential algorithms?

Reminder (LCA)

- RANDOMLY PERMUTE EDGES
- THEN PERFORM GREEDY

$\boxed{\text{LESS DEPENDANCE ON PRIOR-EDGE DECISIONS}}$

(log n)-iteration algorithm

Peeling

$\boxed{n = |V|}$

Applications

- STREAMING
- DISTRIBUTED
- LCA
- MASSIVELY PARALLEL COMPUTATION (MPC)
  (MAPREDUCE, HADOOP, PREGEL)
- DYNAMIC

Peeling algorithm

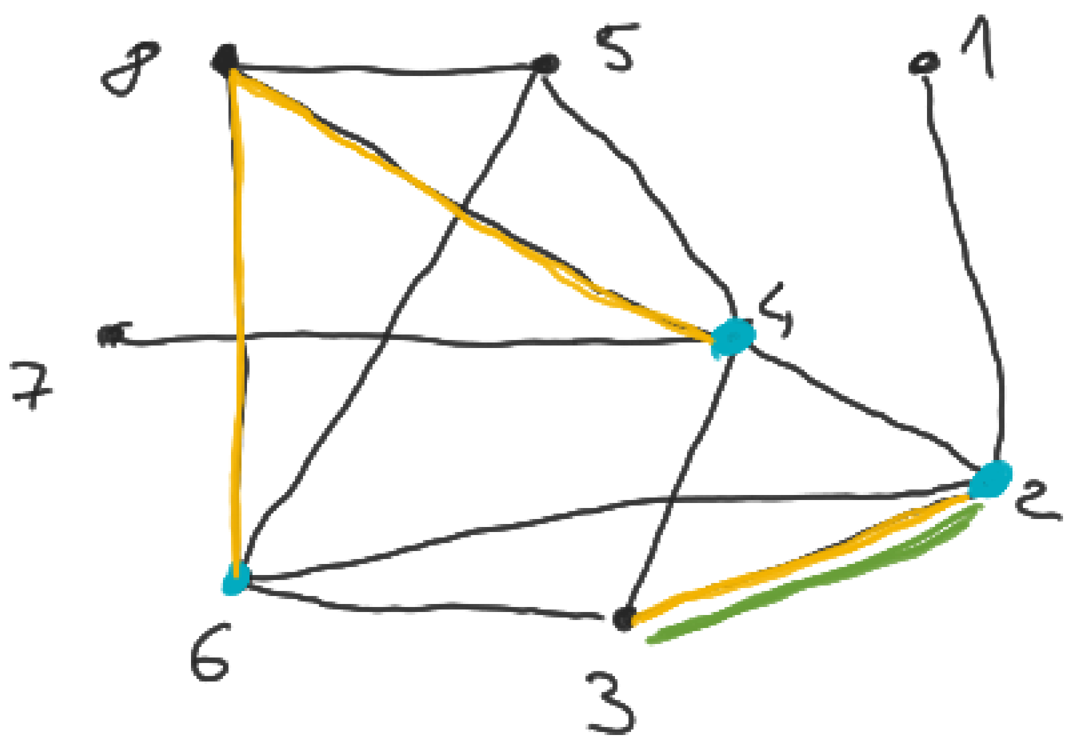
1. $M = \emptyset$

2. For iteration i = 1 ... log n
2.1 Let H_i be the verties having degree at least n/2^i
2.2 For each vertex v in H_i: with prob 9/10 v does nothing; otherwise, w selects an edge incident to it uniformly at random
2.3 Add to M each selected edge disjoint from the other selected ones.
2.4 Remove from the graph H_i and V(M)

3. Return M

H_i = HEAVY VERTICES

M = MATCHING



$H_1 = \{2, 4, 6\}$

ADD TO $M: \{2,3\}$

Peeling algorithm

1. M = ∅

2. For iteration i = 1 ... log n
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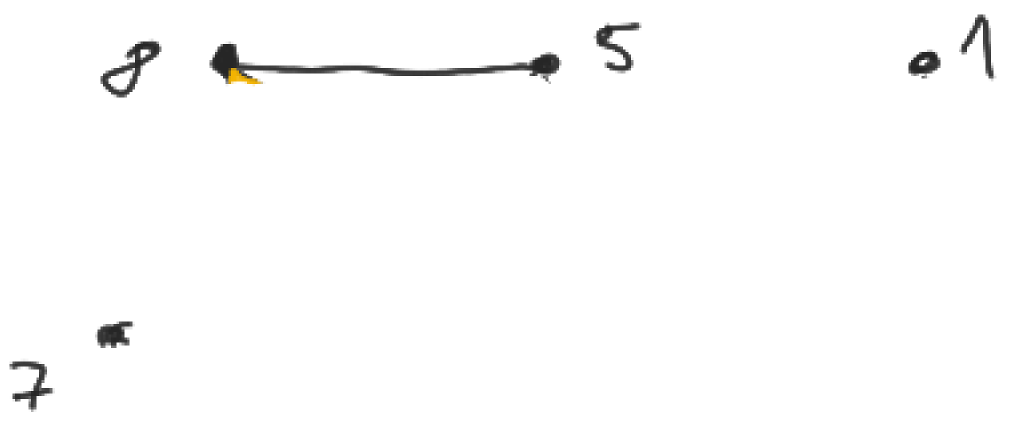

$H_1 = \{2, 4, 6\}$

ADD TO M: {2,3}

Peeling algorithm

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CLAIM:
PEELING CONSTRUCTS A $\Theta(1)$-APPROXIMATE MAXIMUM MATCHING.

Analysis:
Correctness & Complexity

HOW LOCAL PEELING IS?	IS M A MATCHING?

- COLLECTING H_i ✓
- SELECTING EDGES ✓
- RESOLVING CONFLICTS ✓
- ADDING TO M ✓

- BY CONSTRUCTION, EACH EDGE ADDED TO M IS DISJOINT FROM THE OTHER ONES ALREADY IN M. ✓
- SELECTED EDGES ADDED TO M ARE DISJOINT. ✓

Analysis: Approximation

ANALYZE ITERATION i:

R_i = REMOVED VERTICES

M_i = MATCHED VERTICES

H_i = HIGH-DEGREE VERTICES

$$R_i = M_i \cup H_i \implies$$

$$|R_i| \leq |M_i| + |H_i|$$

FIX $w \in H_i$

$$\boxed{\text{WANT: } \Pr[w \in M_i]} = p_w$$

$\{x, w\}$ SELECTED BY w, ASSUMING w PERFORMS STEP 2.2

$$p_w = \Pr[w \text{ PERFORMS STEP 2.2}] \cdot \Pr\left[\begin{array}{l}\text{ONLY } w \text{ POINTS TO } x \text{ \& } \\ x \text{ DOESN'T PERFORM 2.2}\end{array}\right]$$

Analysis: Approximation

(1) $\Pr[w$ PERFORMS STEP 2.2$] = \frac{1}{10}$

(2) $P_x = \Pr[x$ DOESN'T PERFORM 2.2$] \cdot$

$\Pr[$ONLY w POINTS TO $x] \geq$

$\frac{9}{10} \cdot$ [NEXT SLIDE]

FIX $w \in H_i$

$\boxed{\text{WANT: } \Pr[w \in M_i]} = P_w$

$\{x, w\}$ SELECTED BY w, ASSUMING w PERFORMS STEP 2.2

$P_w = \Pr[w$ PERFORMS STEP 2.2$] \cdot$

P_x

$\Pr\left[\begin{array}{l}\text{ONLY } w \text{ POINTS TO } x \text{ \&} \\ x \text{ DOESN'T PERFORM 2.2}\end{array}\right]$

Analysis: Approximation

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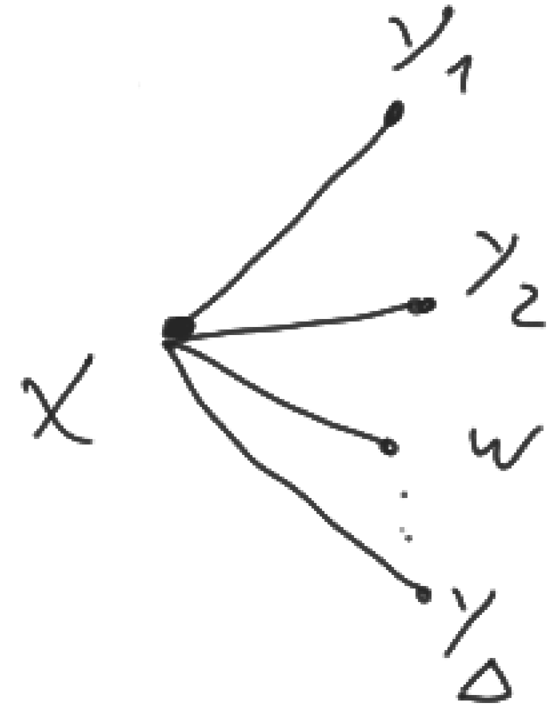
$\Pr[\text{ONLY } w \text{ POINTS TO } x] \geq$

$$\frac{9}{10} \cdot \left(1 - \frac{1}{5\Delta}\right)^{\Delta} \simeq$$

$$\frac{9}{10} \cdot e^{-1/5} \geq 0.7$$



$\boxed{\text{WHEN } y_k \longrightarrow x\,?}$

- $d(y_k) \geq \frac{n}{2^i}$
- y_k PERFORMS STEP 2.2
- y_k SELECTS $\{y_k, x\}$

$$\Pr[y_k \to x] \leq \frac{1}{10} \cdot \frac{2}{\Delta} = \frac{1}{5} \cdot \frac{1}{\Delta}$$

$$\Pr[y_k \not\to x] \geq 1 - \frac{1}{5\Delta}$$

Analysis: Approximation

(1) $\Pr[w \text{ PERFORMS STEP 2.2}] = \frac{1}{10}$

(2) $P_x = \Pr[x \text{ DOESN'T PERFORM 2.2}] \cdot$

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$$\frac{9}{10} \cdot \left(1 - \frac{1}{5\Delta}\right)^{\Delta} \simeq$$

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$$P_w = \frac{1}{10} \cdot \frac{7}{10} = \frac{7}{100} \overset{\text{DEF}}{=} C$$

$$\boxed{\text{A VERTEX FROM } H_i \text{ IS IN } M_i \text{ WITH PROB.} \geq C}$$

$$E\left[\frac{|M_i|}{2}\right] \geq C \cdot E[|H_i|]$$

$$E[|R_i|] \leq E[|M_i|] + E[|H_i|] \leq \left(1 + \frac{1}{2C}\right) E[|M_i|]$$

Analysis: Approximation

$$\sum_i E[|R_i|] \leq \left(1+\frac{1}{2c}\right)\sum_i E[|M_i|]$$

- THE GRAPH AFTER $\log n$ ITERATIONS HAS NO EDGES (SO, $\bigcup_i R_i$ IS A VERTEX COVER)

$\Rightarrow$

$\bigcup_i M_i$ IS A $\Theta(1)$-APPROX. MAXIMUM MATCHING

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