Lecture 18:

Lower bound techniques

How to prove lower bounds?

easy? sublinear time algorithms see very little of input.

difficult? sublinear time algorithms are usually randomized

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Useful lower bound tool:

Yao's Principle: Given distribution D on union of "positive" (Yes, PASS) instances + "negative" (No, FAIL) inputs, such that any deterministic algorithm of query complexity $\leq t$ is incorrect with prob $\geq 1/3$ on inputs chosen from D, then t is a lower bound on randomized query complexity.

(Proof Omitted)

ave case deterministic l.b.

$\Downarrow$

randomized worst case l.b.

Game theoretic view:

Alice selects deterministic alg A
Bob selects input x
$\Big\}$ payoff = cost of $A(x)$

A selects randomized algorithm $\Longleftrightarrow$ A picks random deterministic algorithm
(fixed once you set random bits)

Von Neuman's minimax $\Rightarrow$ when A randomized,
a $\underbrace{\text{randomized Bob}}_{\text{distribution on inputs}}$ can do just as well
as when A deterministic

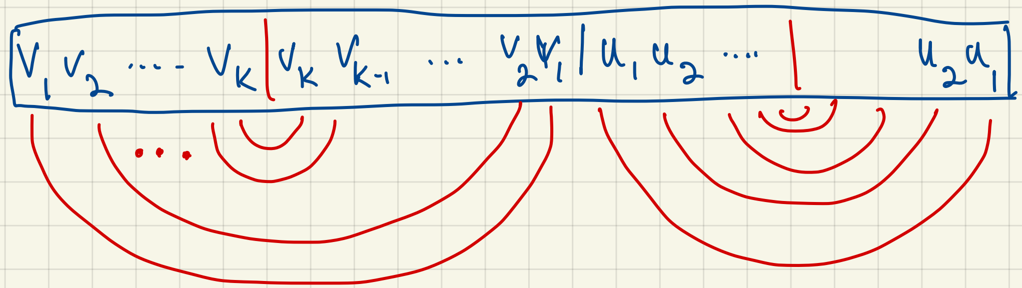
$\Rightarrow$ if want to show lower bnd, need only show distribution on inputs that is bad for every deterministic algorithm

Example application of Yao's method:

$$2PAL = \{ w \mid w \text{ is } w = v v^R u u^R \}$$

Concatenation of palindromes

e.g. $\underbrace{000111}_{v}\,\underbrace{111000}_{v^R}\,\underbrace{011100}_{u}\,\underbrace{001110}_{u^R}$



Note that testing $PAL = \{ w \mid w = v v^R \}$ is trivial:

pick random i, if $w_i \neq w_{n-i}$ fail

Thm any property tester for 2PAL needs $\sqrt{n}$ queries

e.g. if $\mathcal{A}$ satisfies $\forall x \in 2PAL$, $\Pr[\mathcal{A}(x) = PASS] \geq 2/3$

& $\forall x$ ε-far from 2PAL, $\Pr[A(x) = FAIL] \geq 2/3$

then $\mathcal{A}$ makes $\Omega(\sqrt{n})$ queries

Pf. Plan: give distribution on inputs that is hard for all algorithms using $o(\sqrt{n})$ queries.

Yao $\Rightarrow$ randomized l.b. of $\Omega(\sqrt{n})$

Distribution on "Fail" inputs:

F = random string of distance $\geq \varepsilon n$ from 2PAL

Distribution on "Pass" inputs: (wlog assume $6 | n$)

$$P = \begin{cases} 1. & \text{pick } k \in_R \left[\frac{n}{6}+1, \frac{n}{3}\right] \\ 2. & \text{pick random } v, u \text{ s.t. } |v| = k, \ |u| = \frac{n-k}{2} \\ 3. & \text{output } v v^R u u^R \end{cases}$$

note: some strings can be generated via multiple k's

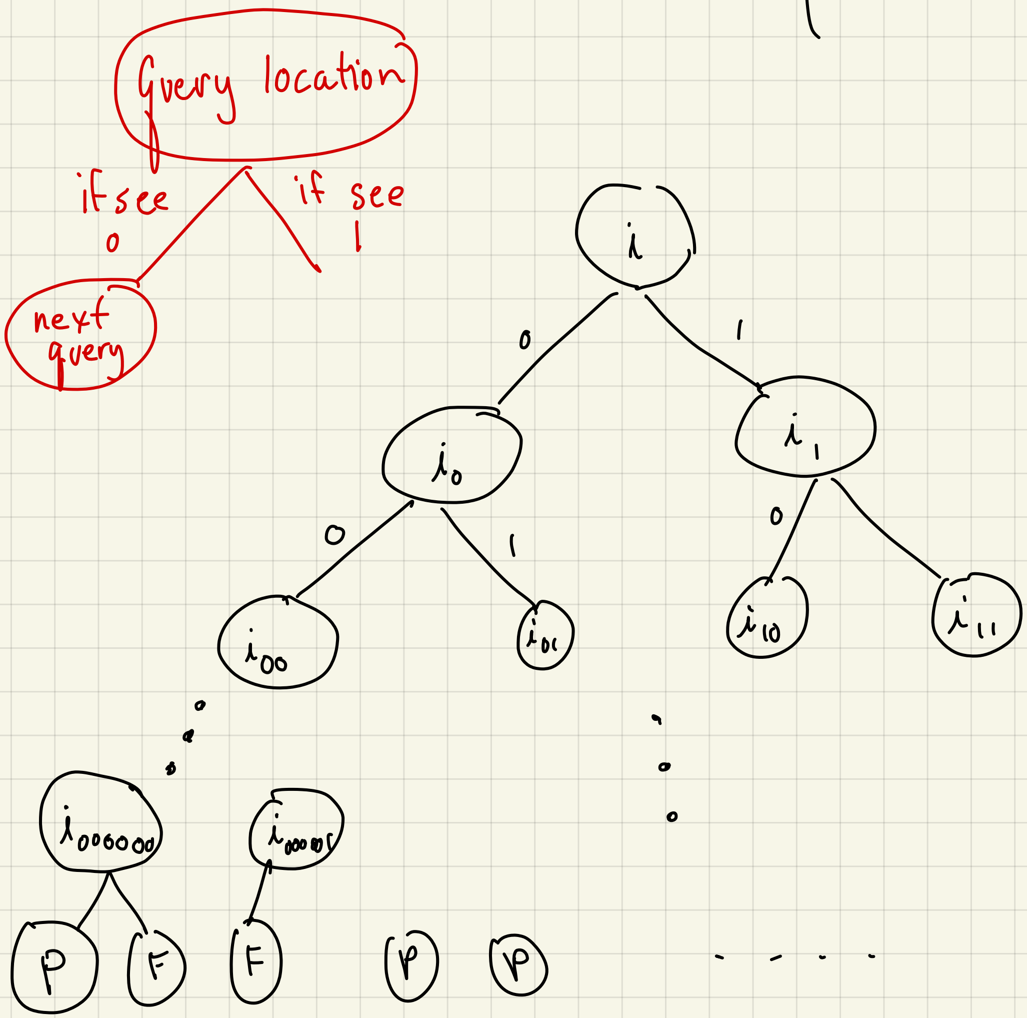
e.g. 0000....0

Bad Distribution:

$$D = \begin{cases} \cdot \ \text{flip coin} \\ \cdot \ \text{if } H \text{ output according to } F \\ \quad \text{else} \quad " \quad " \quad " \quad P \end{cases}$$

Assume deterministic algorithm A s.t. $\begin{cases} \forall\, x \in 2PAL, \quad Pr[A(x) = PASS] \geq 2/3 \\ \& \ \forall\, x \ \varepsilon\text{-far from } 2PAL, \quad Pr[A(x) = FAIL] \geq 2/3 \\ \text{makes } o(\sqrt{n}) \text{ queries} \end{cases}$

describe via query tree:
(for inputs of size n)



hopefully inputs that reach here are supposed to pass

- wlog all leaves have depth t
- $\leq 2^t$ root-leaf paths

We can calculate prob of reaching each leaf given input dist

For each leaf ℓ:

$$E^-(\ell) = \{ \overset{\text{inputs}}{w} \in \{0,1\}^n \text{ s.t. } \underbrace{\text{dist}(w, 2PAL) \geq \varepsilon n}_{w \text{ should FAIL}} \text{ \& } w \text{ reaches leaf } \ell \}$$

$$E^+(\ell) = \{ \overset{\text{inputs}}{w} \in \{0,1\}^n \underbrace{\cap\, 2PAL}_{w \text{ should PASS}} \text{ \& } w \text{ reaches leaf } \ell \}$$

Total error of $\mathcal{A}$ on D:

$$= \sum_{\ell \text{ passing}} \Pr_{w \in D}[w \in \underset{\text{should Fail}}{E^-(\ell)}] + \sum_{\ell \text{ failing}} \Pr_{w \in D}[w \in \underset{\text{should PASS}}{E^+(\ell)}]$$

For each leaf ℓ:

$$E^-(\ell) = \{ \overset{\text{inputs}}{w} \in \{0,1\}^n \text{ s.t. } \underbrace{\text{dist}(w, 2PAL) \geq \varepsilon n}_{w \text{ should FAIL}} \text{ \& } w \text{ reaches leaf } \ell \}$$

$$E^+(\ell) = \{ \overset{\text{inputs}}{w} \in \{0,1\}^n \underbrace{\cap 2PAL}_{w \text{ should PASS}} \text{ \& } w \text{ reaches leaf } \ell \}$$

Total error of A on D:

$$= \sum_{\ell \text{ passing}} \Pr_{w \in D}[\underset{\text{should fail}}{w \in E^-(\ell)}] + \sum_{\ell \text{ failing}} \Pr_{w \in D}[\underset{\text{should PASS}}{w \in E^+(\ell)}]$$

Claim 1 if $t = o(n)$, $\forall \ell$ at depth t

$$\Pr_D[w \in E^-(\ell)] \geq \left(\tfrac{1}{2} - o(1)\right) 2^{-t}$$

} so "Fail" inputs show up at all leaves

Claim 2 if $t = o(\sqrt{n})$, $\forall \ell$ at depth t

$$\Pr_D[w \in E^+(\ell)] \geq \left(\tfrac{1}{2} - o(1)\right) 2^{-t}$$

} so "PASS" inputs show up at all leaves

But each leaf has to choose one label! will be wrong on almost 1/2

Total error of A on D:

$$= \sum_{\ell \text{ passing}} \left(\tfrac{1}{2} - o(1)\right) 2^{-t} + \sum_{\ell \text{ failing}} \left(\tfrac{1}{2} - o(1)\right) 2^{-t} \geq \tfrac{1}{2} - o(1) >> \tfrac{1}{3}$$

$\boxtimes$

Claim 1 if $t = o(n)$, $\forall \ell$ at depth t

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$P = \begin{cases} 1. & \text{pick } k \in_R [\frac{n}{6}+1, \frac{n}{3}] \\ 2. & \text{pick random } v, u \text{ s.t. } |v| = k \quad |u| = \frac{n-k}{2} \\ 3. & \text{output } v v^R u u^R \end{cases}$

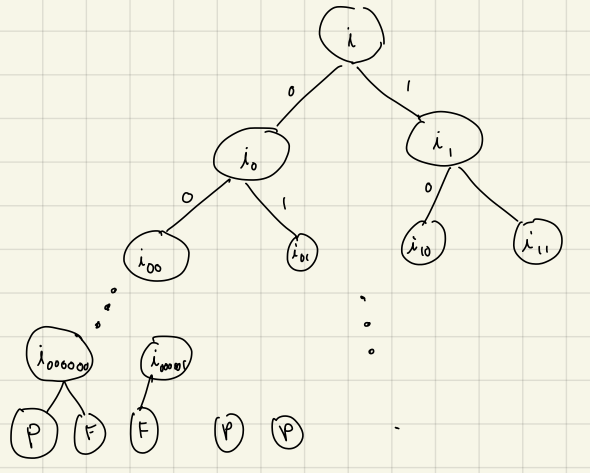
D =
- flip coin
- if H output according to F
  else " " " P

Proof:

Plan:

- F is close to U
- U is uniformly distributed at each leaf (each locn has random bit, so go left/right with equal probability)

$$\Longrightarrow \Pr_{w \in U}[w \in E^-(\ell)] = \frac{2^{n-t}}{2^n} = 2^{-t}$$



But how much can distribution on leaves change using F?

(input size n) $|2PAL_n| \le 2^{\frac{n}{2}} \cdot \frac{n}{2}$ ← choice of x

choice of u, v

$$\#\text{ words at distance } \varepsilon \text{ from } 2PAL_n \le 2^{\frac{n}{2}} \cdot \frac{n}{2} \cdot \sum_{i=0}^{\varepsilon n} \binom{n}{i} \le 2^{n/2 + 2\varepsilon \log(\frac{1}{\varepsilon}) \cdot n}$$

Very few!!

so $\quad E^-(\ell) \geq \underbrace{2^{n-t}}_{\text{\# strings in } \mathcal{U} \text{ reaching } \ell} - \underbrace{2^{\frac{n}{2}+2\varepsilon\log\frac{1}{\varepsilon}n}}_{\text{\# words not in } F} = (1-o(1))2^{n-t}$

- assume $\varepsilon << 1/8$
- t is $o(n)$

so 1st term swamps this

so $\quad \Pr_D[w \in E^-(\ell)] \geq \frac{1}{2}\Pr_F[w \in E^-(\ell)]$

$$= \frac{1}{2}\frac{|E^-(\ell)|}{2^n} \geq \left(\frac{1}{2}-o(1)\right)2^{-t} \qquad \blacksquare$$

Claim 2 if $t = o(\sqrt{n})$, $\forall \ell$ at depth t

$$\Pr_b[w \in E^+(\ell)] \ge \left(\tfrac{1}{2} - o(1)\right)2^{-t}$$

so "PASS" inputs show up at <u>all</u> leaves

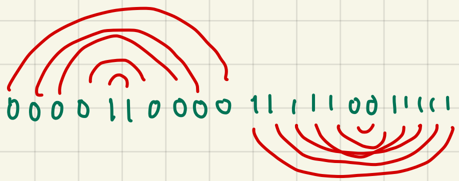
Proof Plan: for every fixed set of $o(\sqrt{n})$ queries, lots of strings in 2PAL follow the path

how many strings agree with leaf ℓ? 2^{n-t}

how many n-bit strings in 2PAL agree with leaf ℓ?

$\ge 2^{\frac{n}{2}-t} - ???$

difficulty:



Fix $k=10$: should see same value at:

1, 10
2, 9
3, 8
⋮

Lots of "dependencies"

F = random string of distance $\ge \varepsilon n$ from 2PAL

$P = \begin{cases} 1. & \text{pick } k \in_R \left[\frac{n}{6}+1, \frac{n}{3}\right] \\ 2. & \text{pick random } v, u \text{ s.t. } |v| = k \quad |u| = \frac{n-k}{2} \\ 3. & \text{output } vv^R uu^R \end{cases}$

D =
- flip coin
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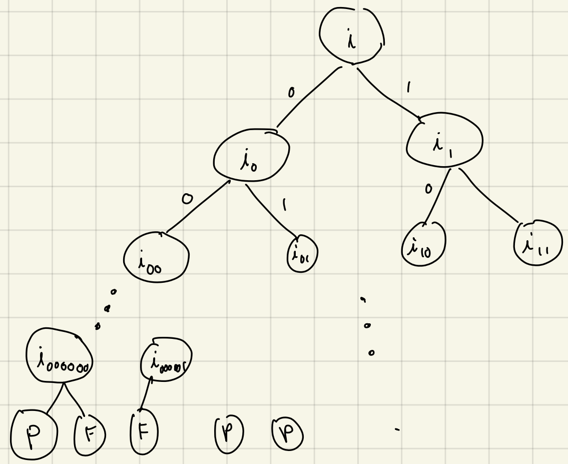
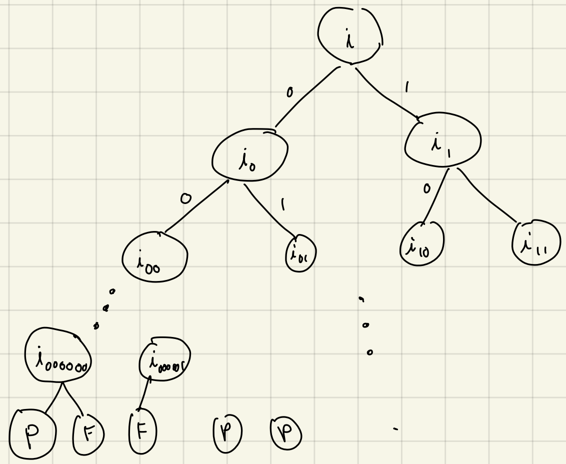


Maybe <u>no</u> string follows path?

but K is picked randomly! in $\left[\frac{n}{6}+1, \dots, \frac{n}{3}\right]$

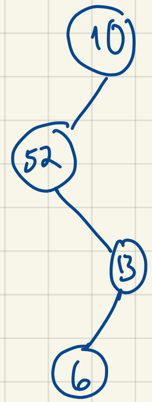
hope: paths that pair up dependent queries for one K will do badly on most others?

$$P = \begin{cases} 1. & \text{pick } K \in_R \left[\frac{n}{6}+1, \frac{n}{3}\right] \\ 2. & \text{pick random } v, u \text{ s.t. } |v| = k \quad |u| = \frac{n-k}{2} \\ 3. & \text{output } vv^R uu^R \end{cases}$$

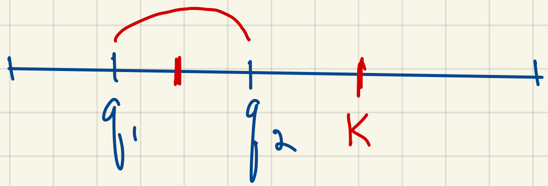


Consider leaf ℓ,

$Q_\ell \leftarrow$ indices queried along way



$\forall$ pair $q_1, q_2 \in Q_\ell$, at most 2 choices of K "pair" them:



only 1 choice in this case

$\Rightarrow$ # choices of K s.t. $\underline{\underline{\text{no pair}}}$ in Q_ℓ $\underbrace{\text{symmetric around } k \text{ or } \frac{n}{2}+k}_{\text{"Good" } K}$

$$\geq \frac{n}{6} - 2\binom{t}{2} = (1 - o(1))\left(\frac{n}{6}\right)$$

Claim 2 if $t = o(\sqrt{n})$, $\forall \ell$ at depth t

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Proof

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\# choices of k s.t. no pair in Q_ℓ $\underbrace{\text{symmetric around } k \text{ or } \tfrac{n}{2}+k}_{\text{"Good" } k}$

$$\ge \frac{n}{6} - 2\binom{t}{2} = (1-o(1))\left(\frac{n}{6}\right)$$

So $$\Pr_P[w \in E^+(\ell)] = \sum_w \sum_k \underbrace{\Pr_P[w|k]}_{2^{-n/2}} \cdot \underbrace{\Pr[\text{choose } k]}_{6/n} \cdot 1_{w \in E^+(\ell)}$$

$$\ge \frac{1}{\left(\frac{n}{6}\right)\left(2^{n/2}\right)} \cdot \left[(1-o(1))\frac{n}{6}\right] 2^{\frac{n}{2}-t} = (1-o(1))\,2^{-t}$$

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