Lecture 18:

Lower bound techniques

How to prove lower bounds?

easy? sublinear time algorithms see very little of input.

difficult? sublinear time algorithms are usually randomized

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Useful lower bound tool:

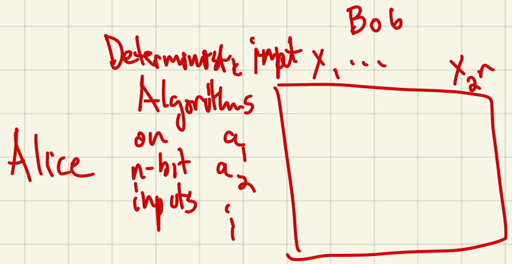
average case lower bound

$\Downarrow$

randomized worst case l.b.

Yao's ~~Principle~~ Thm.: Given distribution D on union of "positive" (Yes, PASS) instances + "negative" (No, FAIL) inputs, such that any deterministic algorithm of query complexity $\leq t$ is incorrect with prob $\geq 1/3$ on inputs chosen from D, then t is a lower bound on randomized query complexity.

(Proof omitted) (see Wikipedia)



Game theoretic view:

Alice selects deterministic alg A
Bob selects input x
$\Big\}$ payoff $=$ cost of $A(x)$

A selects randomized algorithm $\Longleftrightarrow$ A picks random deterministic algorithm (includes random bits)

Von Neuman's minimax $\Rightarrow$ when A randomized, a $\underbrace{\text{randomized}}_{\text{distribution on inputs}}$ Bob can do just as well as when A deterministic

$\Rightarrow$ if want to show l.b. need only show a "bad" distribution on inputs that is "hard" for any deterministic algorithm

Example application of Yao's method:

$2PAL = \{ w \mid w \text{ is } w = vv^R uu^R \}$ Concatenation of 2 palindromes

e.g. $\overbrace{000111}^{v}\,\overbrace{111000}^{v^R}\,\overbrace{011100}^{u}\,\overbrace{001110}^{u^R}$

$\boxed{V_1 V_2 \cdots V_K \mid V_K V_{K-1} \cdots V_2 V_1 \mid u_1 u_2 \cdots \mid \quad u_2 u_1}$

Note that testing $PAL = \{ w \mid w = vv^R \}$ is "easy"

pick random i, if $w_i \neq w_{n-i}$ FAIL

Can test 2PAL in $O(\sqrt{n})$ time

Can you do better?

Thm any property tester for 2PAL needs $\sqrt{n}$ queries

e.g. if $\mathcal{A}$ satisfies $\forall x \in 2PAL$, $\Pr[A(x) = PASS] \geq 2/3$,

& $\forall x$ ε-far from 2PAL, $\Pr[A(x) = FAIL] \geq 2/3$

then A makes $\Omega(\sqrt{n})$ queries

Pf. Plan: give distribution on inputs that is hard for all deterministic algorithms using $o(\sqrt{n})$ queries.

Yao $\Rightarrow$ randomized l.b. of $\Omega(\sqrt{n})$

Distribution on "Fail" inputs:

F = random string of distance $\geq \varepsilon n$ from 2PAL

Distribution on "Pass" inputs: (wlog assume $6 | n$)

$$P = \begin{cases} 1. & \text{pick } k \in_R \left[\frac{n}{6}+1, \frac{n}{3}\right] \\ 2. & \text{pick random } v, u \text{ s.t. } |v| = k,\ |u| = \frac{n-k}{2} \\ 3. & \text{output } vv^R uu^R \end{cases}$$

note: some strings can be generated by multiple k's

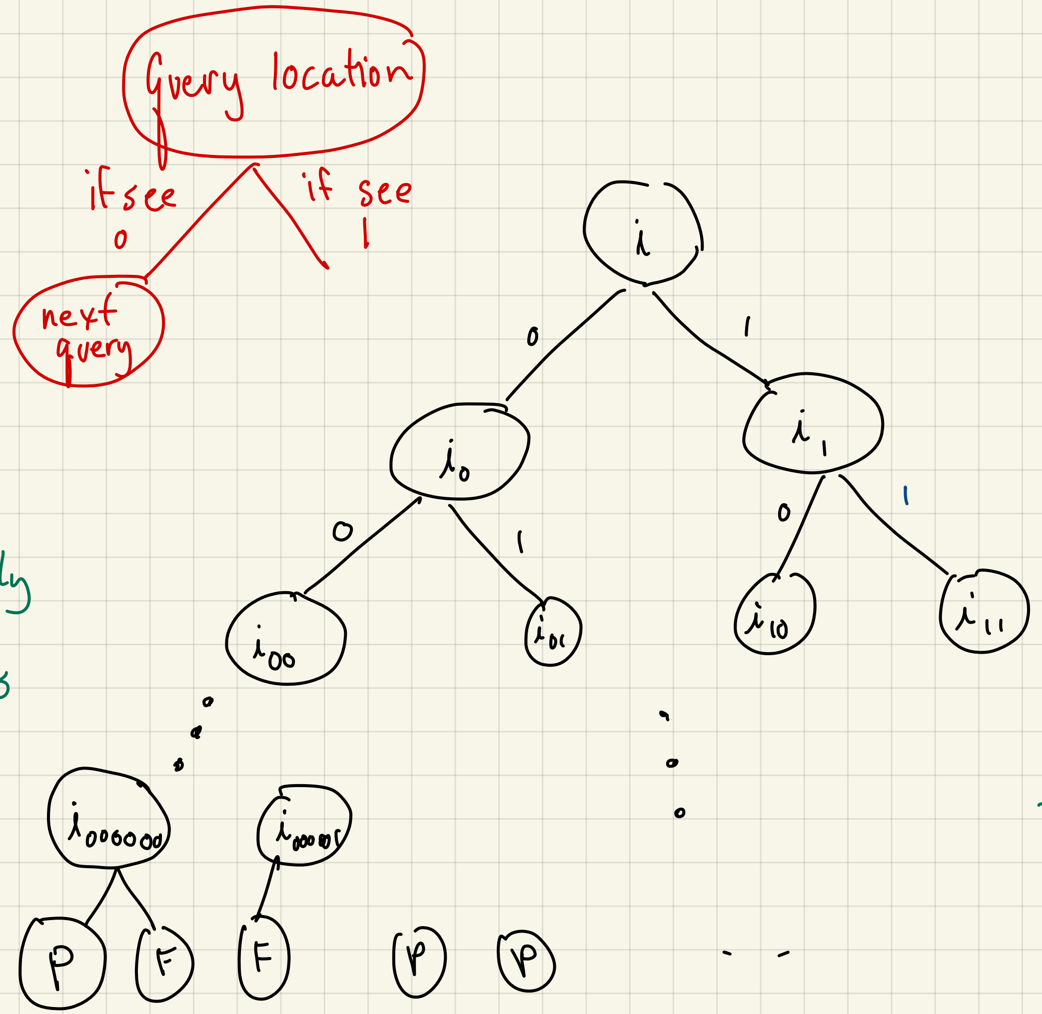
e.g. $111 \cdots 1$

Bad Distribution:

$$D = \begin{cases} \text{flip coin} \\ H: \text{ output according to } F \\ T: \text{ output according to } P \end{cases}$$

Assume deterministic algorithm A

describe via query tree: (for inputs of size n)



assume no "repeats" along path

- depth of decisiontree is t
- wlog assume all leaves have depth t (complete binary tree)
- 2^t root-leaf paths

each input follows exactly one branch
reaches leaf, which is hopefully labelled by correct answer

we can calculate prob of reaching each leaf given (input dist F_p)

Suppose $w \in_R \{0,1\}^n$: $\Pr[w \text{ reaches leaf } \ell] = 2^{-t}$

For each leaf ℓ:

$$E^-(\ell) = \{ \overset{\text{inputs}}{w} \in \{0,1\}^n \quad \text{s.t.} \quad \underbrace{\mathrm{dist}(w, 2PAL) \geq \varepsilon n}_{\text{w should Fail}} \ \text{\& } w \text{ reaches leaf } \ell \}$$

$$E^+(\ell) = \{ \overset{\text{inputs}}{w} \in \{0,1\}^n \underbrace{\cap\, 2PAL}_{\text{w should PASS}} \quad \text{\&} \quad w \text{ reaches leaf } \ell \}$$

Total error of $\mathcal{A}$ on $\mathcal{D}$:

$$= \sum_{\ell \ \underbrace{\text{passing}}_{\text{output of leaf}}} \Pr_{w \in \mathcal{D}} [w \in \underbrace{E^-(\ell)}_{\text{correct answer is Fail}}] + \sum_{\ell \ \text{failing}} \Pr_{w \in \mathcal{D}} [w \in \underbrace{E^+(\ell)}_{\text{should PASS}}]$$

For each leaf ℓ:

$E^-(\ell) = \{ \overset{\text{inputs}}{w} \in \{0,1\}^n$ s.t. $\underbrace{\text{dist}(w, 2PAL) \geq \varepsilon n}_{w \text{ should FAIL}}$ & w reaches leaf $\ell \}$

$E^+(\ell) = \{ \overset{\text{inputs}}{w} \in \{0,1\}^n \underbrace{\cap\, 2PAL}_{w \text{ should PASS}}$ & w reaches leaf $\ell \}$

Total error of $\mathcal{A}$ on D:

$$= \sum_{\ell \text{ passing}} \Pr_{w \in D}[\underbrace{w \in E^-(\ell)}_{\text{should Fail}}] + \sum_{\ell \text{ failing}} \Pr_{w \in D}[\underbrace{w \in E^+(\ell)}_{\text{should PASS}}]$$

Claim 1 if $t = o(n)$, $\forall \ell$ at depth t

$$\Pr_D[w \in E^-(\ell)] \geq \underbrace{\left(\tfrac{1}{2} - o(1)\right)}_{\text{almost } 1/2 \text{ prob that random inputs reach } \ell} 2^{-t}$$

} so FAIL inputs show up at all leaves

Claim 2 if $t = o(\sqrt{n})$, $\forall \ell$ at depth t

$$\Pr_D[w \in E^+(\ell)] \geq \left(\tfrac{1}{2} - o(1)\right) 2^{-t}$$

} so PASS inputs show up at all leaves

But each leaf has to choose a label so will be wrong on almost 1/2 inputs that reach it

Total error of $\mathcal{A}$ on D:

$$= \sum_{\ell \text{ passing}} \left(\tfrac{1}{2} - o(1)\right) 2^{-t} + \sum_{\ell \text{ failing}} \left(\tfrac{1}{2} - o(1)\right) 2^{-t} \geq \tfrac{1}{2} - o(1) >> \tfrac{1}{3}$$

$\blacksquare$

Claim 1 if $t = o(n)$, $\forall \ell$ at depth t

$$\Pr_D[\, w \in E^-(\ell)] \geq \left(\tfrac{1}{2} - o(1)\right) 2^{-t}$$

So "Fail" inputs show up at all leaves

F = random string of distance $\geq \varepsilon n$ from 2PAL

$$P = \begin{cases} 1. & \text{pick } k \in_R \left[\frac{n}{6}+1, \frac{n}{3}\right] \\ 2. & \text{pick random } v, u \text{ s.t. } |v| = k \quad |u| = \frac{n-k}{2} \\ 3. & \text{output } v v^R u u^R \end{cases}$$

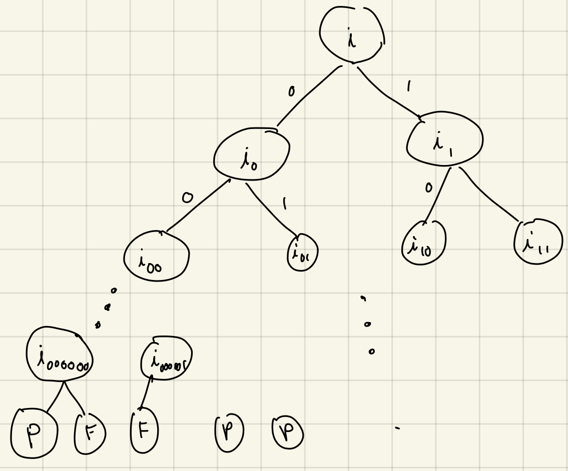
$$D = \begin{cases} \cdot & \text{flip coin} \\ \cdot & \text{if } H \text{ output according to } F \\ & \text{else} \quad \text{" " " } P \end{cases}$$

Proof:

Plan:

- F is close to U
- U is uniformly distributed at each leaf (each locn has random bit, so go left/right with equal probability)

$$\Rightarrow \Pr_{w \in U}[\, w \in E^-(\ell) \cup E^+(\ell)] = \frac{2^{n-t}}{2^n} = 2^{-t}$$



But how much can distribution on leaves change using F?

$$|2PAL_n| \leq \underbrace{\frac{n}{2}}_{\text{choice of } i} \cdot \underbrace{2^{n/2}}_{\text{choice of } u, v}$$

$$\#\text{ words at dist } \leq \varepsilon n \text{ from 2PAL} \leq \left(2^{n/2} \cdot \frac{n}{2}\right) \sum_{i=0}^{\varepsilon n} \binom{n}{i} \leq 2^{n/2 + 2\varepsilon \log \frac{1}{\varepsilon} \cdot n} \quad \text{very few}$$

so $E^-(\ell) \geq 2^{n-t} - 2^{\frac{n}{2} + 2\varepsilon \log \frac{1}{\varepsilon} \cdot n} = (1 - o(1))\, 2^{n-t}$ since $t << \frac{n}{2}$

\# strings in u that reach ℓ

\# words not in F

assume $\varepsilon << 1/8$, t is $o(n)$ } so 1st term swamps this

so $\Pr_D[w \in E^-(\ell)] \geq \frac{1}{2} \cdot \Pr_F[w \in E^-(\ell)]$

$= \frac{1}{2} \frac{|E^-(\ell)|}{2^n} \geq \left(\frac{1}{2} - o(1)\right) 2^{-t}$

Claim 2 if $t = o(\sqrt{n})$, $\forall \ell$ at depth t

$$\Pr_b[w \in E^+(\ell)] \geq \left(\tfrac{1}{2} - o(1)\right)2^{-t}$$

so "PASS" inputs show up at all leaves

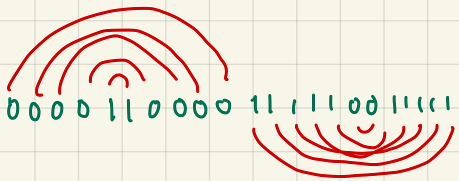
Proof Plan: for every fixed set of $o(\sqrt{n})$ queries, lots of strings in 2PAL follow the path

how many strings agree with leaf ℓ? 2^{n-t}

how many n-bit strings in 2PAL agree with leaf ℓ?

$\geq 2^{\frac{n}{2} - ??} - ??$

difficulty:



Fix $K = 10$: should see same value at

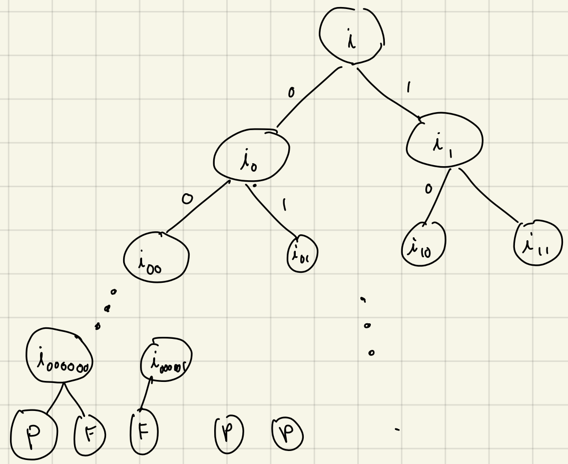
1, 10
2, 9
3, 8
⋮

Lots of dependencies

F = random string of distance $\geq \varepsilon n$ from 2PAL

P =
1. pick $K \in_R [\frac{n}{6}+1, \frac{n}{3}]$
2. pick random v, u s.t. $|v| = k$, $|u| = \frac{n-k}{2}$
3. output $vv^R uu^R$

D =
- flip coin
- if H output according to F
- else " " " P

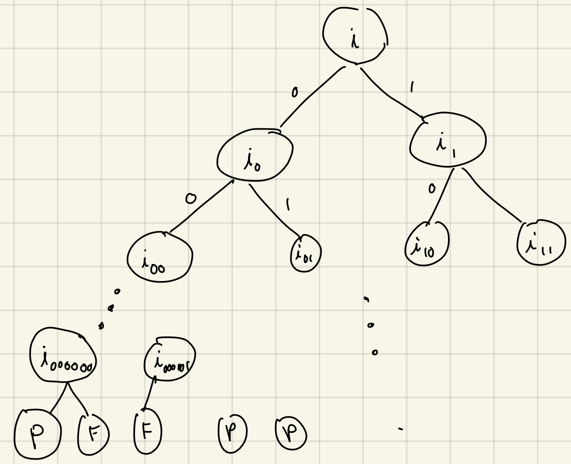


Maybe no string in 2PAL follows the path?

but k is picked randomly! in $[\frac{n}{6}+1, \dots, \frac{n}{3}]$

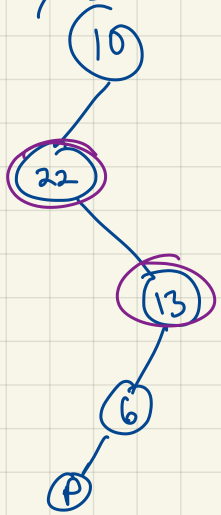
$P = \begin{cases} 1. & \text{pick } k \in_R [\frac{n}{6}+1, \frac{n}{3}] \\ 2. & \text{pick random } v, u \text{ s.t. } |v| = k \quad |u| = \frac{n-k}{2} \\ 3. & \text{output } vv^R uu^R \end{cases}$

hope: paths that pair up dependent queries for one k will do badly on most others?



Consider leaf ℓ,

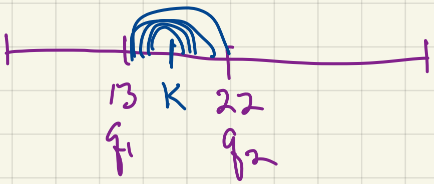
$Q_\ell = \{10, 22, 13, 6\}$



$Q_\ell \leftarrow$ indices queried along way

$\forall$ pair $q_1, q_2 \in Q_\ell$, at most 2 choices of k "pair" them!

1?



if p picked k that pairs $q_1 + q_2$ then all bets off

$\Rightarrow$ # choices of k s.t. no pair in Q_ℓ $\underbrace{\text{symmetric around } k \text{ or } \frac{n}{2}+k}_{\text{good } k}$

$\geq \frac{n}{6} - 2 \cdot \binom{t}{2} = (1 - o(1))\frac{n}{6}$

1?

Claim 2 if $t = o(\sqrt{n})$, $\forall \ell$ at depth t

$$\Pr_D[w \in E^+(\ell)] \geq \left(\tfrac{1}{2} - o(1)\right)2^{-t}$$

so "PASS" inputs show up at all leaves

Proof

Plan: for every fixed set of $o(\sqrt{n})$ queries, lots of strings in 2PAL follow the path

how many strings agree with leaf ℓ? 2^{n-t}

how many n-bit strings in 2PAL agree with leaf ℓ?

$$\geq 2^{\frac{n}{2}-t} - ???$$

\# choices of k s.t. no pair in Q_ℓ $\underbrace{\text{symmetric around } k \text{ or } \tfrac{n}{2}+k}_{\text{"Good" } k}$

$$\geq \frac{n}{6} - 2\binom{t}{2} = (1-o(1))\left(\frac{n}{6}\right)$$

So
$$\Pr_P[w \in E^+(\ell)] = \sum_w \sum_k \underbrace{\Pr_P[w \mid k]}_{2^{-n/2}} \cdot \underbrace{\Pr[\text{choose } k]}_{(n/6)^{-1}} \cdot 1_{w \in E^+(\ell)}$$

$$\geq \frac{1}{\cancel{\left(\frac{n}{6}\right)}\cancel{2^{n/2}}} \left[(1-o(1))\cancel{\frac{n}{6}}\right] \cdot \cancel{2^{\frac{n}{2}}}^{-t} = (1-o(1))2^{-t}$$

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