Toward Unfolding Doubly Covered n-Stars

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Open Problem 22.3 from [1] asks: does every closed polyhedron P have a general unfolding to a non-overlapping polygon? A *general unfolding* is produced by cutting the surface along the edges of a cut tree spanning the vertices of P and flattening it to a connected, planar piece without overlap (here the cuts are not restricted to the edges of the polyhedron). It is known that convex polyhedra always admit general unfoldings [2]. This is also true for non-convex but nearly flat polyhedra [4] and for various classes of orthogonal polyhedra where the cuts are restricted to be parallel to the polyhedron edges [3]. At CCCG in August 2017, Stefan Langerman posed the question of finding general unfoldings for doubly covered polygons, and more specifically for doubly covered *n-stars*: regular n-gons with identical isosceles triangular "spikes" attached to their edges (all spikes have the same base angle $\alpha \in (0, \pi/2)$). Does every doubly covered n-star admit a general unfolding?

In this paper, we explore the space of doubly covered n-stars in search of families of general unfoldings. We show that general unfoldings of doubly covered n-stars exist for:

- any base angle $\alpha \in (0, \pi/2)$, for $n \in \{3 \ldots 10, 12\}$
- any n, for base angle $\alpha < \frac{\pi}{3}\left(\frac{n+3}{n}\right)$.

We prove existence by construction, providing families of general unfoldings within specific subdomains of n and α. Figure 1 shows representative constructions from some relevant families.

References

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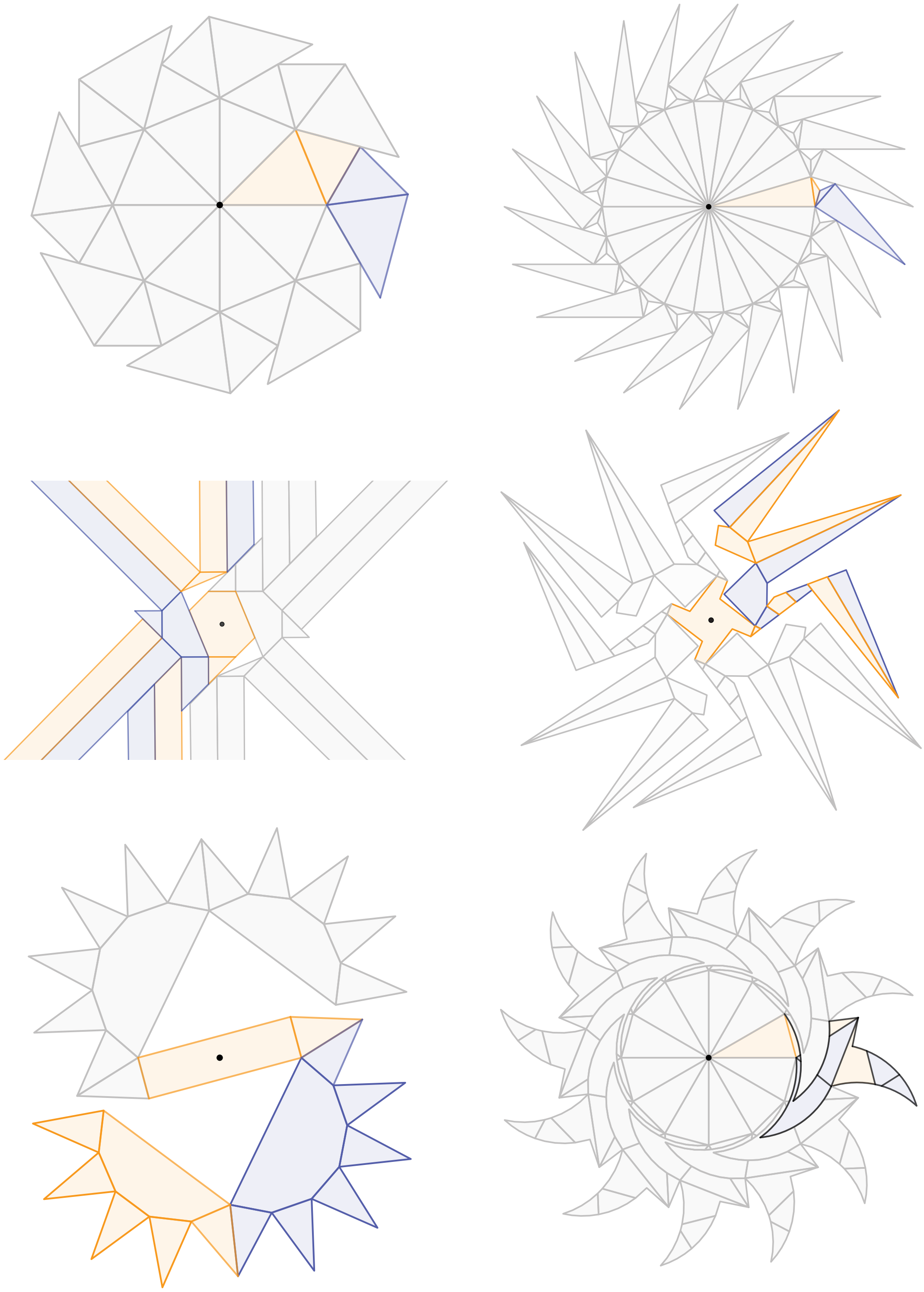

Figure 1: Top row: naively unfolding each spike provides valid unfoldings up to $\alpha \leq \frac{\pi}{6}\left(\frac{n+6}{n}\right)$ for $n \leq 12$ (left shows $n = 8$), and up to $\alpha \leq \pi\left(\frac{3}{n}\right)$ for $12 < n$ (right shows $n = 22$); cuts are made along edges only; representative top and bottom surface pieces are color-coded. Middle row: unfolding families for $n = 8$ (left) and $n = 12$ (right) for any $\alpha \in (0, \pi/2)$; the latter case uses a more complex cut tree to avoid overlap. Bottom row: improvements over the naive unfoldings for $\alpha \leq \frac{\pi}{3}\left(\frac{n+2}{n}\right)$ for even n (left), and for $\alpha < \frac{\pi}{3}\left(\frac{n+3}{n}\right)$ for all n (right); while the right unfolding supersedes the left, both families are constrained by the same asymptotic upper bound $\alpha < \pi/3$, in the limit of large n.