Folding Polyominoes with Holes into a Cube

Oswin Aichholzer* Hugo A. Akitaya† Kenneth C. Cheung‡ Erik D. Demaine§ Martin L. Demaine§
Sándor P. Fekete¶ Linda Kleist¶ Irina Kostitsyna‖ Maarten Löffler** Zuzana Masárová††
Klara Mundilova‡‡ Christiane Schmidt♣

Abstract

When can a polyomino piece of paper be folded into a unit cube? Prior work studied tree-like polyominoes, but polyominoes with holes remain an intriguing open problem. We present sufficient conditions for a polyomino with hole(s) to fold into a cube, and conditions under which cube folding is impossible. In particular, we show that all but five special *simple* holes guarantee foldability.

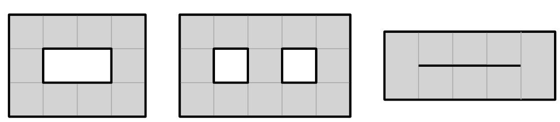

Figure 1: Three polyominoes that fold along grid lines into a unit cube, from puzzles by Nikolai Beluhov [4].

1 Introduction

Given a piece of paper in the shape of a polyomino (i.e., a polygon in the plane formed by unit squares on the square lattice that are connected edge-to-edge), does it have a folded state in the shape of a unit cube? The standard rules of origami apply; in particular, we allow each unit square face to be covered by multiple layers of paper. Examples of this decision problem are given by the three puzzles by Nikolai Beluhov [4] shown in Figure 1. We encourage the reader to print out the puzzles and try folding them.

Prior work [2] studied this decision problem extensively, introducing and solving several different models of folding. This gave rise to a model that matches the puzzles in Figure 1: Fold only along grid lines of the polyomino; allow only orthogonal folding angles ($\pm 90°$ and $\pm 180°$); and forbid folding material strictly interior to the cube. In this model, the prior work [2] characterizes which tree-shaped polyominoes lying within a $3 \times n$ strip can fold into a unit cube.

Notably, however, the polyominoes in Figure 1 are not tree-shaped or even simple: One puzzle has a hole, another puzzle has two holes, and a third puzzle has a degenerate hole (a slit). Arguably, these holes are what makes the puzzles fun and challenging. Therefore, in this paper, we embark on characterizing which polyominoes with hole(s) fold into a unit cube in this model. Although we do not obtain a complete characterization, we give many interesting conditions under which a polyomino does or does not fold into a unit cube.

The problem is sensitive to the choice of model. In the more flexible model allowing half-grid folds and 45° diagonal folds between grid points, the prior work [2] shows that *all* polyominoes of at least ten unit squares can fold into a unit cube, and lists all smaller polyominoes that fold into a cube. Thus this model already has a complete characterization of polyominoes that fold into a cube, including those with holes. Therefore, we focus on the grid-fold model described above.

Specific to polyominoes and polycubes, there is extensive work in this model on finding polyominoes that fold into many different polycubes [3] and into multiple different boxes [1, 5, 6, 7, 8].

Our Results.

1. We identify which polyominoes with a single hole are foldable; see Theorem 1, Section 3.1. In fact, all but five *simple* holes already guarantee foldability.
2. We identify combinations of two (of the remaining five) holes that allow the polyomino to fold into a cube; see Section 3.2.
3. We show that certain of the remaining five simple holes or their combinations do not allow a foldable polyomino; see Section 4.

*Institute for Software Technology, Graz University of Technology, oaich@ist.tugraz.at
†Department of Computer Science, Tufts University, Hugo.Alves_Akitaya@tufts.edu
‡NASA Ames Research Center, kenny@nasa.gov
§CSAIL, Massachusetts Institute of Technology, {edemaine,mdemaine}@mit.edu
¶Department of Computer Science, TU Braunschweig, s.fekete@tu-bs.de, kleist@ibr.cs.tu-bs.de
‖Mathematics and Computer Science Department, TU Eindhoven, i.kostitsyna@tue.nl
**Department of Information and Computing Science, Universiteit Utrecht, m.loffler@uu.nl
††IST Austria, Klosterneuburg, zuzana.masarova@ist.ac.at
‡‡Institute of Discrete Mathematics and Geometry, TU Wien, klara.mundilova@tuwien.ac.at
♣Department of Science and Technology, Linköping University, christiane.schmidt@liu.se

4. We present an algorithm that checks a necessary local condition for foldability; see Section 4.4.

2 Notation

A *polyomino* is a polygon P in the plane formed by a union of $|P| = n$ *unit squares* on the square lattice that are connected edge-to-edge. We do not require a connection between every pair of adjacent squares; that is, we allow *slits* along the edges of the lattice subject to the condition that the polyomino is connected.

We call a set h of connected missing squares and slits a *hole* if the dual has a cycle containing h in its interior. We call a hole of a polyomino *simple* if it is one of the following: a unit square, a slit of size 1, slits of size 2 (corner or straight), or a U-slit of size 3, see Figure 2 for an illustration.

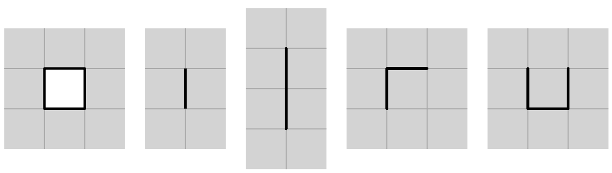

Figure 2: The five simple holes.

A connected three-dimensional polyhedron formed by a union of *unit cubes* on the cubic lattice that are connected face-to-face is called a *polycube*. If the polycube Q is a *unit cube*, we denote it by $Q = \mathcal{C}$.

In this paper, we study the problem of folding a given polyomino P with holes to form $\mathcal{C}$, allowing only 90° and 180° folds along the lattice. We illustrate mountain folds in red, and valley folds in blue. Whenever we show numbers on faces in crease patterns these refer to a "real" die, i.e., opposite faces sum up to 7.

3 Polyominoes That Do Fold

In this section, we present polyominoes that fold. We start with polyominoes that contain a hole guaranteeing foldability.

3.1 Polyominoes with Single Holes

In this section, we show that all holes different from a simple hole guarantee foldability.

Theorem 1 *If a polyomino P contains a hole h that is not simple, then P folds into a cube.*

Proof. It is easy to see that because the hole h is non-simple, it must be a superset of one of the holes in Figure 4, that is, we distinguish the cases where h contains

- Two unit squares sharing an edge
- Two unit squares sharing a vertex

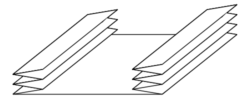

Figure 3: Folding strategy to reduce to seven cases.

- A unit square and an incident slit
- A slit of length at least 3 (straight, zigzagged, L-shaped, or T-shaped)

In a first step, we show that if h contains one of the four above holes, we may assume that it contains exactly this hole. Let h be a hole containing a hole h' of the above type. By definition of a hole, h needs to be enclosed by solid squares. Thus we can sequentially fold the squares of P in columns to the left and right of h' on top of the columns directly left and right of h', respectively, as illustrated in Figure 3. Afterwards, we fold the squares of P in rows to the top and bottom of h' on top of the rows directly top and bottom of h', respectively. We call the resulting polyomino P'. Note that because h is a hole of P, all neighbouring squares of h' exist in P'. Thus we may assume that we are given one of the seven polyominoes depicted in Figure 4, where striped squares may or may not be present.

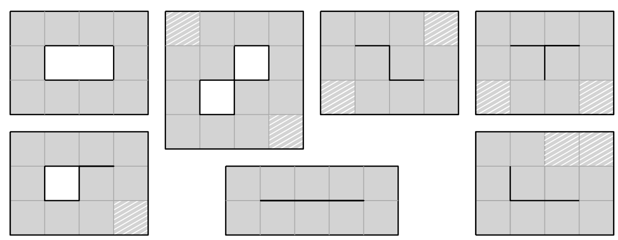

Figure 4: Any polyomino with a hole that is not simple can be reduced to one of the seven illustrated cases; striped squares may or may not be present.

Secondly, we present folding strategies. Note that the case if h' consists of two squares sharing only a vertex, we can fold the top row on its neighboring row and obtain the case where h' consist of a square and an incident slit. For an illustration of the folding strategies for the remaining cases consider Figure 5. □

Are simple holes ever helpful? In fact, four of the five simple holes sometimes allow foldability, as illustrated in Figure 6. Note that the U-slit of size 3 reduces to the square hole. In Lemma 11, we show that the slit of size 1 never helps to fold a rectangular polyomino. Lemma 7 implies that the polyominoes without the holes cannot be folded.

3.2 Combinations of Two Simple Holes

In this section we consider combinations of two simple holes that fold.

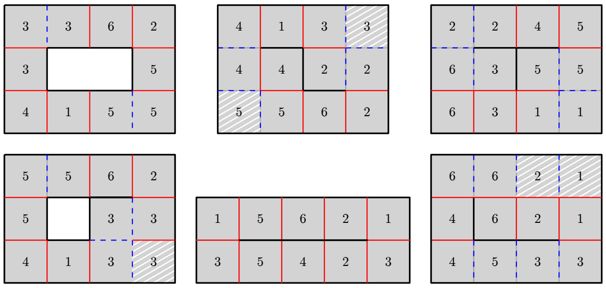

Figure 5: Crease pattern of cube foldings; mountain folds (solid red), valley folds (dashed blue). Squares with the same number cover the same face of the cube.

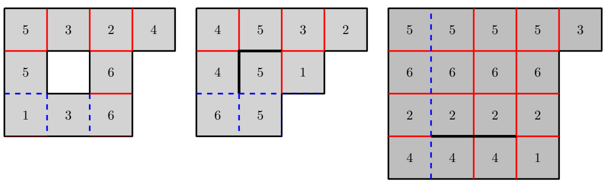

Figure 6: Four simple holes may be helpful.

Theorem 2 *A polyomino with two vertical straight size-2 slits with at least two columns and an odd number of rows between them folds.*

Proof. As in the previous section, we first fold all rows between the slits together to one row; this is possible because there is an odd number of rows between the slits. Then, all the surrounding rows and columns are folded towards the slits. Finally, we fold the columns between the slits to reduce their number to two or three. Depending on whether the number of columns between the slits was even or odd, this yields a shape similar to the one shown in Figure 7 (a) and (b), respectively. Striped squares may be (partially) present. In all cases, the two shapes fold as indicated by the illustrated crease pattern. Note that in Figure 7 (b) the polyomino is of course connected, which implies that for sure at least one square of the central column is part of the polyomino, i.e., a square with label 6 is used. □

If the two slits have only one or no column between them, then the shape cannot be folded as can be verified by the algorithm of Section 4.4.

In the following theorems we call a U-slit which has the open part at the bottom an A-slit. If the orientation of the U-slit is not relevant, then we call it a C-slit.

Theorem 3 *A polyomino with an A-slit and a unit square hole/C-slit in the same column above it, having an even number of rows between them, folds.*

Proof. We can assume that the upper hole is a unit square, as the flaps generated by a C-slit can always be folded away. Similar to before we fold away all surrounding rows and columns and reduce the number of rows between the A-slit and the unit square hole to two. This yields the shape of Figure 7 (c), which can be folded. □

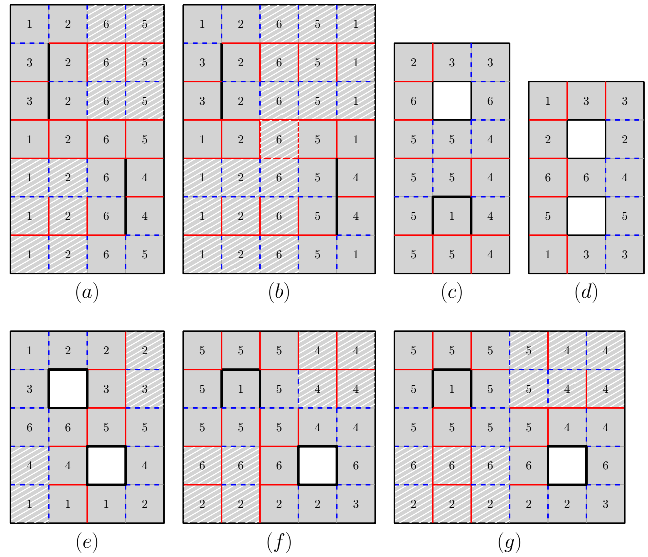

Figure 7: Combinations of two simple holes that are foldable with and without (part of) the striped region.

Note that if the bottom slit is a U-slit, then the shape of Figure 7 (c) does not fold, again verified by the algorithm of Section 4.4.

Theorem 4 *A polyomino with an A-slit and a unit square hole/C-slit below it and separated by an odd number of rows, folds, regardless in which columns they are.*

Proof. As before we assume that the lower hole is a unit square, fold away all surrounding rows and columns, and reduce the number of rows between the two slits/holes to one. Furthermore we fold the columns between the slits/holes to minimize their number. In this way the number of columns between the two slits/holes is at most two, and we obtain one of the shapes shown in Figure 7 (d) to (g). All of them fold, with or without the striped region. Note that the upper unit square holes in Figure 7 (d) and (e) can be replaced by an A-slit which can be folded away. □

Note that if the two holes are in the same or neighboured column(s) (Figure 7 (d) and (e)), then it does not matter which orientation the U-slits have or whether they are unit square holes—any combination folds. We thus get the following statement.

Theorem 5 *A polyomino with two unit square holes which are in the same or in neighboured column(s) and have an odd number of rows between them folds.*

4 Polyominoes That Do Not Fold

In this section, we identify simple holes and simple hole combinations that do not allow the polyomino to fold.

First, we present some results that show how the paper is constrained around an interior vertex.

Lemma 6 *Four faces around a polyomino vertex v for which the dual graph is connected cannot cover more than three faces of $\mathcal{C}$.*

Proof. v is incident to 4 faces in P. As vertices of P are mapped to vertices of $\mathcal{C}$ and all vertices of $\mathcal{C}$ are incident to 3 faces, v is incident to only 3 faces in $\mathcal{C}$. □

Lemma 7 *Four faces around a vertex v not in the boundary of P cannot cover more than two faces of $\mathcal{C}$. In particular, at least two collinear incident creases must be folded by $180°$.*

Proof. Let A, B, C, and D be the faces around v in circular order. By Lemma 6, at most three faces of $\mathcal{C}$ are covered by A, B, C, and D. Hence, at least two faces map to the same face of $\mathcal{C}$. These can either be edge-adjacent of diagonal. For the first case, let this w.l.o.g. be A and B. Hence, the crease between them must be folded by 180°. Then C and D must also map to the same face of $\mathcal{C}$ to maintain the paper connected. The crease between C and D must also be folded by 180°. In the latter case, w.l.o.g. A and C map to the same face of $\mathcal{C}$. As they are both incident to v, only two options of folding those two faces on top of each other exist. Either the edge between A and B gets folded on top of the edge between B and C, this leaves a diagonal fold on B, a contradiction, or the edge between A and D gets folded on top of the edge between B and C, which results in D being mapped to C, and those are two adjacent faces, in which case we already argued that two collinear incident creases must be folded by 180°. □

Corollary 1 *Let v, w be two vertices in P's interior, which share a horizontal edge. If we fold horizontally through v, i.e., if the two collinear incident creases of v folded by $180°$ are horizontal, then we also have to fold $180°$ horizontally through w.*

4.1 Polyominoes with Unit Square Holes, L-Shaped Holes and U-Shaped Holes

We begin by examining the possible foldings of a polyomino containing a unit square hole. Suppose that a given polyomino P with a unit square hole h folds into a cube. Furthermore, let the shape of h no longer be a square in the folded state. That is, the hole h is folded in a *non-trivial* way. Then, in the folded state, either all edges of h are glued together, or two pairs of edges are glued forming an L-shape. We will argue that if P folds into a cube, the first case is impossible, while the second produces a specific crease pattern around h.

Lemma 8 *The four edges of a unit square hole of a polyomino that folds into a cube cannot be all glued together in the folded state.*

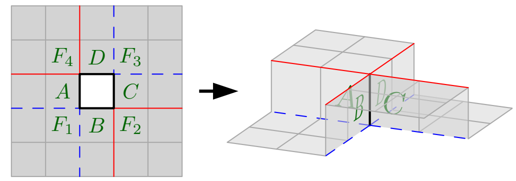

Figure 8: Four edges of a square hole glued together.

Proof. Let the four faces of the polyomino edge-adjacent to the hole be A, B, C, and D, and the four faces vertex-adjacent to the hole be F_1, F_2, F_3, and F_4, see Figure 8. Consider A, F_1, and B in the folded state. As the two corresponding edges of the hole are glued together, the three faces must be pairwise perpendicular. The similar statement holds for the triples $\{B, F_2, C\}$, $\{C, F_3, D\}$, and $\{D, F_4, A\}$. Therefore, if P folds into a cube, face A must be glued to face C, face B must be glued to D, F_1 to F_3, and F_2 to F_4. Suppose, w.l.o.g., in the folded state face A lies in a more outer layer than C. Then, F_1 and F_4 are in a more outer layer than F_3 and F_2, respectively. Thus, face B connects the more inner layer of F_2 to the more outer layer of F_1, and at the same time D connects the inner layer of F_3 to the outer layer of F_4. Hence, faces B and D intersect, which is impossible. Therefore, if the polyomino folds into a cube, the four edges of a square hole cannot all be glued together. □

It follows that the only non-trivial way to glue the edges of a square hole of a polyomino folded into a cube is to form an L-shape. This effectively amounts to gluing a pair of diagonal vertices of the hole.

Let a, b, c, and d be the four vertices of the hole, and suppose a and c are glued together when folding the polyomino into a cube, see also Figure 9 (left). Consider the crease pattern around the hole. We shall only focus on the angles of the creases and not the type of the fold, as there may be (and will be) other creases in P affecting the type of the creases under our consideration. Observe that the three faces incident to each of the vertices b and d are pairwise perpendicular, they form a corner of a

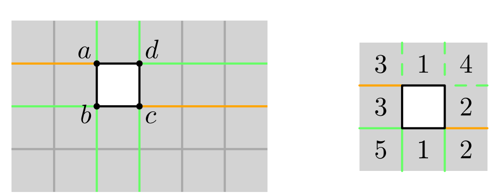

Figure 9: Left: crease pattern around a hole folding into an L-shape when gluing vertices a and c; 90° creases are shown in green, and 180° creases in orange. Right: numbers indicate the face of the cube in the folded state; mountain folds are shown in solid, and valley folds as dashed lines.

cube. Thus, the creases emanating from b and d are all $90°$. Further observe that the three faces around each of the vertices a and c fold into two faces of a cube, thus leading to one of the creases being $90°$ and the other $180°$. Finally, the two $180°$ creases are parallel to each other. Indeed, consider the right side of Figure 8. For a crease to form an L-shape one of the two dashed blue lines must fold to $180°$, which corresponds to two parallel creases in the unfolded state. Therefore, the crease pattern in Figure 9 (left) is the only pattern of creases (up to rotation and reflection) around a non-trivially folded square hole. Figure 9 (right) shows the faces of the corresponding crease pattern.

Note that the arguments above extend to an L-shape slit of size 2, and a U-slit of size 3.

Theorem 9 *Two holes, which are either unit square, L-slit of size* 2*, or U-slit of size* 3*, of a polyomino* P *such that (1)* P *contains all the other cells of the bounding box of the two holes, (2)* P *folds into a cube, cannot be both folded non-trivially if the number of rows and the number of columns between the holes is at least* 1*.*

Proof. It follows from the above observations that if there were two unit square holes, both folded non-trivially, with positive number of rows and columns between them, there would be two intersecting $90°$ creases. □

Theorem 10 *A rectangle with two unit square holes in the same row does not fold into a cube if the number of columns between the holes is even.*

Proof (sketch). We prove that a 3×6 rectangle with two unit squares holes as in Figure 10 does not fold into a cube. From that it follows that any $3 \times (4 + 2k)$ rectangle with two unit square holes in the same row separated by $2k$ columns does not fold into a cube. Note that both holes must be folded non-trivially, otherwise the polyomino cannot be folded into a cube.

The vertical fold in the middle of two holes must be a $180°$ fold as depicted in Figure 10; otherwise there would be two perpendicular $90°$ creases. There are two types of crease patterns for this polyomino: when pairs of parallel $90°$ creases run vertical, and when there is one pair of horizontal parallel $90°$ creases. In both cases, the faces in between of those creases all map to the same face on $\mathcal{C}$, which implies that the face opposite to the one on $\mathcal{C}$ cannot be covered. □

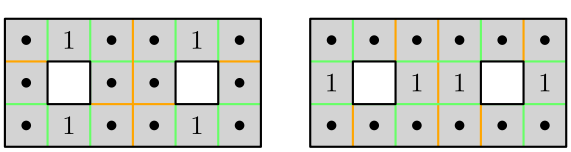

Figure 10: A polyomino that does not fold into a cube.

4.2 Polyominoes with a Single Slit of Size 1

The following Lemma shows that slit holes of size one do not help in folding a rectangular polyomino into $\mathcal{C}$.

Lemma 11 *A rectangular polyomino* P *with a single slit hole of size 1 does not fold into* $\mathcal{C}$*.*

Proof. Because of Corollary 1 we can restrict to the polyomino in Figure 11. Let A, B, C, D, E and F be the faces adjacent to h as in Figure 11. Because the paper must remain connected, the endpoints of h must map to adjacent vertices of $\mathcal{C}$. Then the paper behaves exactly as if the slit were not there as follows. If E and B maps to the same face of $\mathcal{C}$, then A (resp., C) must map to the same face as F (resp., D). Otherwise, E and B maps to adjacent faces. Then, A and C (resp., F and D) maps to the same face as B (resp., E). By the successive application of Lemma 7 in a rectangular polyomino P, without loss of generality only the front, left, back and right faces of $\mathcal{C}$ can be covered. □

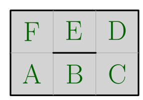

Figure 11: A polyomino with a slit hole of size one.

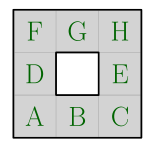

Figure 12: A polyomino with a single square hole.

4.3 Rectangles with a Single Square Hole

In this section, we show the following fact:

Theorem 12 *If* P *is a rectangle with a single square hole* h*, then* P *does not fold into a unit cube* $\mathcal{C}$*.*

Proof. Let P' denote the 3×3 rectangle with a central unit square hole depicted in Figure 12. By Corollary 1 any polyomino P needs to be reduced to P':

Claim 1 *Every rectangle with a single unit square hole is foldable (if and) only if* P' *is foldable.*

Consequently, it remains to show that

Claim 2 *The polyomino* P' *does not fold into* $\mathcal{C}$*.*

We label the eight faces of P' by (A,B,C,D,E,F,G,H) as depicted in Figure 12. Without loss of generality assume that A maps to the top face of $\mathcal{C}$. First, we argue that C cannot map to the same face. If that was true, then B also maps to the top face and by the number of faces, every remaining face must map to a different face of $\mathcal{C}$. However, if D maps to the back face of $\mathcal{C}$, so does E, a contradiction. Consequently, A and C do not map to the same face. By symmetry, F does not map to the top face (and neither C nor F map to the same face as H).

Next, we argue that the only faces that can map to the bottom face of $\mathcal{C}$ are C and F: If E would map to the bottom face, any of (B,C) or (D,F,G,H) must cover the front face and right face, respectively. For B to cover the front face, C must cover the bottom face. (Analogously, the argument for G.) H has odd number of squares to A, if H would map to the bottom face, one face would have to be between A and H, hence, we would need to reduce the number by folding, this folds H onto C or F. A contradiction to the first fact. Hence, only faces that can map to the bottom face of $\mathcal{C}$ are C and F.

W.l.o.g., let F be the bottom face, and D the back face. Then the only faces that can cover the left face are C and H; in particular, if E covers the left face then the right face remains uncovered. Thus, we assume w.l.o.g. that C covers the left face. Hence, B maps to the top face. Now, if E maps to the back face, both G and H must map to the right face of $\mathcal{C}$, and the front face is uncovered. If E maps to the left face, because the top and left faces are doubly covered, every remaining face must be singly covered. Then, H must map to the front face. But face G can only map to the top face, which cannot happen because A and B already cover this face. The only remaining case is when both C and F map to the bottom face, thus, B and D maps to the right and back faces of $\mathcal{C}$ respectively. However, both E and G can only cover faces of $\mathcal{C}$ that are already covered (bottom, back and right faces), and $\mathcal{C}$ would not have all its faces covered. □

4.4 An Algorithm to Check a Necessary Local Condition for Foldability

Consider the following local condition: let s be a square in a polyomino P such that the mapping between vertices of s and vertices of a face of $\mathcal{C}$ has been fixed. Then, for every adjacent square s' of s, there are two possibilities how to map its four vertices onto $\mathcal{C}$: the two vertices shared by s and s' must be mapped consistently and for the other two vertices there are two options depending on whether s' is folded at 90° angle to an adjacent face of $\mathcal{C}$, or whether it is folded at 180° to the same face of $\mathcal{C}$.

The algorithm below checks whether there exists a mapping between all vertices of squares of P to vertices of $\mathcal{C}$ such that the above condition holds for every pair of adjacent polyomino squares of P.

1. Run a breadth-first-search on the polyomino squares, starting with the leftmost square in the top row of P and continue via adjacent squares. This produces a numbering of polyomino squares in which each but the first square is adjacent to at least one square with smaller number.
2. Map vertices of the first square to the bottom face of $\mathcal{C}$. Extend the mapping one square at a time according to the numbering, respecting the local condition (that is, in up to two ways). Track all such partial mappings.

The algorithm is exponential, because unless inconsistencies are produced, the number of possible partial mappings doubles with every polyomino square. Nevertheless, it can be used to show non-foldability for small polyominoes: if no consistent mapping exists for a polyomino, then the polyomino cannot be folded onto $\mathcal{C}$. On the other hand, any consistent vertex mapping covering all faces of $\mathcal{C}$ obtained by the algorithm that we tried could in practice be turned into a folding. However, we have not been able to prove that this is always the case.

The algorithm above was used to prove that polyominoes in Figure 13 do not fold, as well as it aided us to find the foldings of polyominoes in Figure 7. An implementation of the algorithm is available at `http://github.com/zuzana-masarova/cube-folding`.

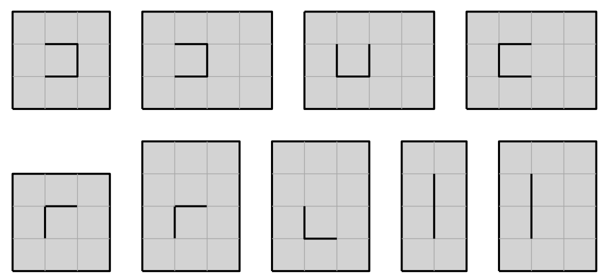

Figure 13: These polyominoes with single L, U and straight size-2 slits do not fold.

5 Conclusion

We showed that, if a polyomino P does contain a non-simple hole, then P folds into $\mathcal{C}$. Moreover, we showed that a unit square hole, size 2 slits (straight or corner), and a size-3 U-slit sometimes allow for foldability.

Based on the presented results, we created a font of 26 polyominoes with slits that look like each letter of the alphabet, and each fold into $\mathcal{C}$. See Figure A in the appendix, and `http://erikdemaine.org/fonts/cubefolding/` for a web app.

We conclude with a list of interesting open problems:

- Does a consistent vertex mapping output by the algorithm in Section 4.4 imply that the polyomino is foldable? If so, is the folding uniquely determined?
- Is any rectangular polyomino with one L-shape, U-shape or straight size-2 slit foldable? Currently, we only know that the small polyominoes in Figure 13 do not fold.

6 Acknowledgments

This research was performed in part at the 33rd Bellairs Winter Workshop on Computational Geometry. We thank all other participants for a fruitful atmosphere.

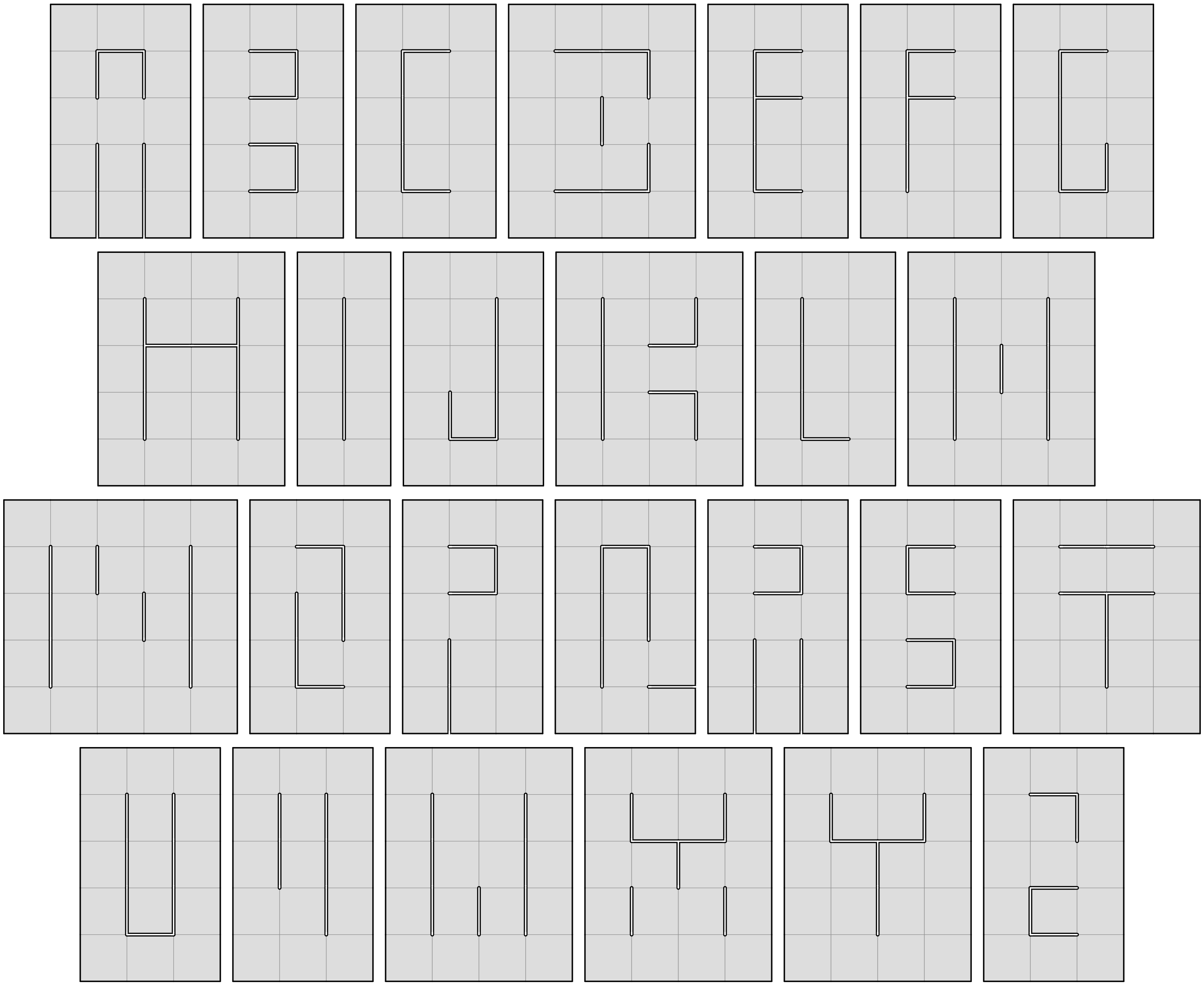

Figure A: Cube-folding font: the slits representing each letter enable each rectangular puzzle to fold into a cube.

References

[1] Z. Abel, E. Demaine, M. Demaine, H. Matsui, G. Rote, and R. Uehara. Common developments of several different orthogonal boxes. In *Proc. 23rd Can. Conf. Comp. Geom. (CCCG)*, 2011. Paper 49.

[2] O. Aichholzer, M. Biro, E. D. Demaine, M. L. Demaine, D. Eppstein, S. P. Fekete, A. Hesterberg, I. Kostitsyna, and C. Schmidt. Folding polyominoes into (poly)cubes. *Int. J. Comp. Geom. & Applic.*, 28(03):197–226, 2018.

[3] G. Aloupis, P. K. Bose, S. Collette, E. D. Demaine, M. L. Demaine, K. Douïeb, V. Dujmović, J. Iacono, S. Langerman, and P. Morin. Common unfoldings of polyominoes and polycubes. In *Proc. 9th Int. Conf. Comp. Geom., Graphs & Appl. (CGGA)*, volume 7033 of *LNCS*, pages 44–54, 2011.

[4] N. Beluhov. Cube folding. `https://nbpuzzles.wordpress.com/2014/06/08/cube-folding/`, 2014.

[5] J. Mitani and R. Uehara. Polygons folding to plural incongruent orthogonal boxes. In *Proc. 20th Can. Conf. Comp. Geom. (CCCG)*, 2008.

[6] T. Shirakawa and R. Uehara. Common developments of three incongruent orthogonal boxes. *Int. J. Comp. Geom. & Applic.*, 23(1):65–71, 2013.

[7] R. Uehara. A survey and recent results about common developments of two or more boxes. In *Origami^6: Proc. 6th Int. Meeting Origami in Sci., Math. and Educ. (OSME 2014)*, volume 1, pages 77–84. American Mathematical Society, 2014.

[8] D. Xu, T. Horiyama, T. Shirakawa, and R. Uehara. Common developments of three incongruent boxes of area 30. In R. Jain, S. Jain, and F. Stephan, editors, *Proc. 12th Ann. Conf. Theory and Applic. of Models of Comput. (TAMC)*, pages 236–247, 2015.