LECT 05

Note Title 2/21/2007

TODAY

- Bounded depth circuits : AC^0
- Parity & AC^0 (mod "Switching Lemma")
- Proof of Switching Lemma.

WARNING: THESE NOTES CONTAIN ONLY THE PROOF OF THE LEMMA;

No statement;

No context!

Furst-Saxe-Sipser Switching Lemma

- Let f be a DNF formula with s wires, on n-inputs.

Let ρ be a random restriction of $x_1 \dots x_n$

$$x_i \leftarrow 0 \quad \text{w.p. } \frac{1-p}{2}$$

$$x_i \leftarrow 1 \quad \text{w.p. } \frac{1-p}{2}$$

$$x_i \leftarrow x_i \quad \text{w.p. } p$$

- Then if $p < \_\_ f(s,n)$

$f|_\rho$ is a function of C inputs.

ind. of n

Proof: Do the restriction in 2 stages

first stage $x_i \leftarrow x_i$ w.p. $\sqrt{p}$.

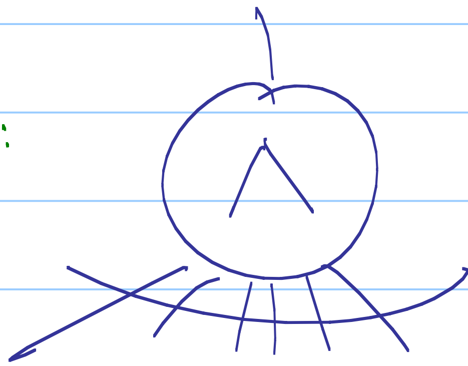
Second stage $x_i \leftarrow x_i$ w.p. $\sqrt{p}$.

Claims:

- First Stage: All terms have $\leq c$ variables
- Second Stage: f depends on $\leq C_0$ variables

Proof of first Stage Claim: (relatively easy)

Case 1:



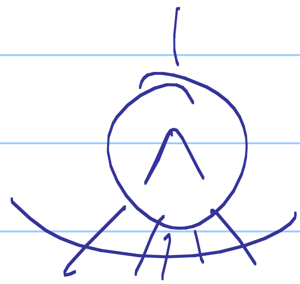
fanout large $\geq 2\log s$

$$\Pr[\text{output} \neq 0] \leq \left(\frac{1}{2} + \sqrt{p}\right)^{2\log s}$$

$$\leq \frac{1}{s^2}$$

$$\Pr[\exists \text{ and gate with output} \neq 0] \leq \frac{1}{s}$$

Case 2:



fanout $\leq 2\log s$

$$\Pr[\exists\, i \text{ unrestricted gate}] \leq (2\log s)^i \cdot \left(\frac{1}{\sqrt{p}}\right)^i$$

So ... if $(\sqrt{p})^i \ll \frac{1}{S}$ then ....

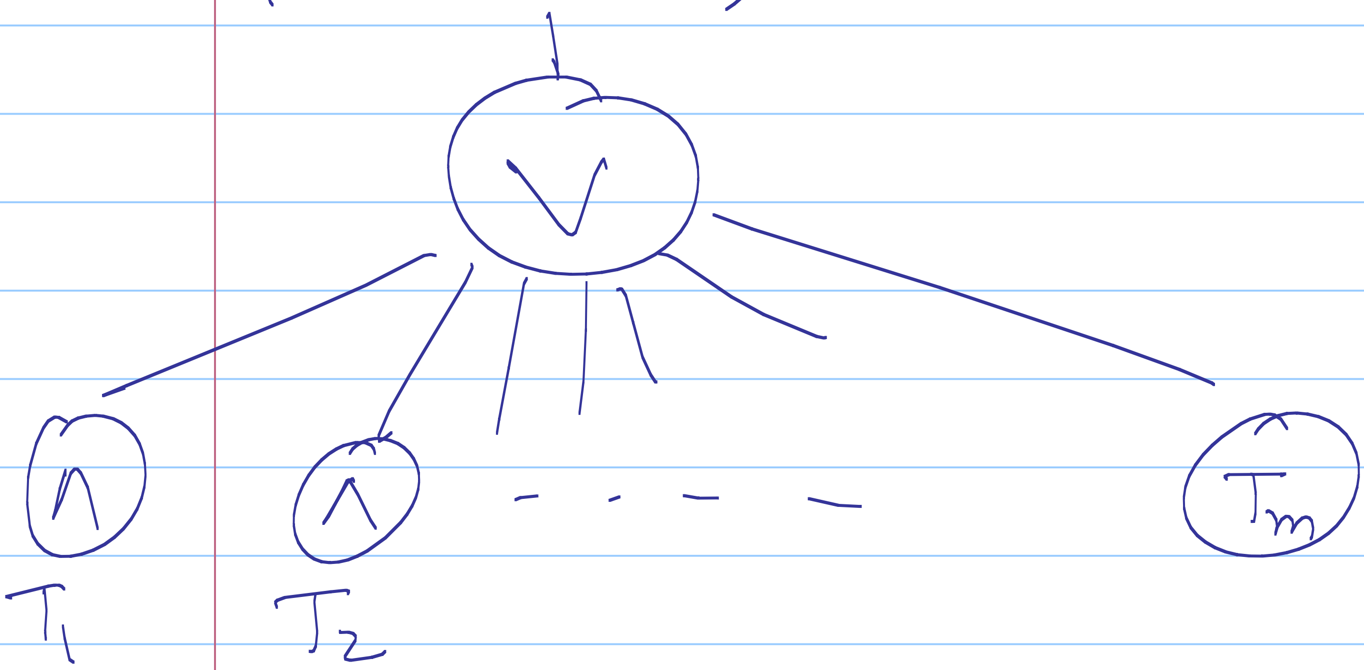
Moral : c - depends on S vs. $\frac{1}{p}$

if $\delta = p^{-a}$ $\quad c \approx 2a$.

Proof of Stage 2:

- More complicated: Induction on c (from first stage)

- Why? Terms from first level overlap (share variables)



Case 1 : Many disjoint T_i's . $(\Omega(3^c \log s))$

$$\Pr[T_i = 1] \geq \left(\frac{1}{2} - \frac{1}{\sqrt{p}}\right)^c$$

$$\geq \frac{1}{3^c}$$

$$\Pr[\forall i\, T_i \neq 1] \leq \left(1 - \frac{1}{3^c}\right)^{\Omega(3^c \log s)}$$

$$\leq \frac{1}{s}$$

$\boxtimes$

Case 2 : (The hard, inductive, case)

Less than $3^c \log s$ disjoint T_i's

Let $T_1 \ldots T_k$ be maximal disjoint set.

Let $H = \bigcup_{i=1}^{k} T_i$

$$[n] = H \cup X$$

$$[n]\big|_\rho = H' \cup X'$$

Want to show: $f\big|_\rho (H' \cup X')$ depends on b_c variables

Step 1: $|H'|$ is small (as in first stage)

Step 2: $\forall$ assignment ρ' to H'

$f\big|_{\rho \cup \rho'} (X')$ depends on few variables

Induction: every term depends on only $c-1$ variables

$\Rightarrow$ depends on $\leq b_{c-1}$ vars.

Step 3: $f|_{\rho \vee \rho'}(H^{\rho} \vee x')$ depends

on $D_c \leq C' + 2^{c'} D_{c-1}$ variables $\boxtimes$