DS-210: PROGRAMMING FOR DATA SCIENCE

LECTURE 29

1. GRAPH EXPLORATION OVERVIEW
2. BREADTH–FIRST SEARCH (BFS)
3. DEPTH–FIRST SEARCH (DFS)
4. BONUS CONTENT: STRONGLY CONNECTED COMPONENTS

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GRAPH EXPLORATION

Sample popular methods:

- breadth–first search (BFS)
  - uses a queue

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GRAPH EXPLORATION

Sample popular methods:

- breadth–first search (BFS)
  - uses a queue
- depth–first search (DFS)
  - uses a stack
- random walks
  - example: PageRank (see Homework 10)

USEFUL GRAPH SUBROUTINES

In [2]:

```
type Vertex = usize;
type ListOfEdges = Vec<(Vertex,Vertex)>;
type AdjacencyLists = Vec<Vec<Vertex>>;

#[derive(Debug)]
struct Graph {
    n: usize, // vertex labels in {0,...,n-1}
    outedges: AdjacencyLists,
}

// reverse direction of edges on a list
fn reverse_edges(list:&ListOfEdges)
        -> ListOfEdges {
    let mut new_list = vec![];
    for (u,v) in list {
        new_list.push((*v,*u));
    }
    new_list
}

reverse_edges(&vec![(3,2),(1,1),(0,100),(100,0)])
```

Out[2]:

```
[(2, 3), (1, 1), (100, 0), (0, 100)]
```

USEFUL GRAPH SUBROUTINES

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Out[2]:

```
[(2, 3), (1, 1), (100, 0), (0, 100)]
```

In [3]:

```
impl Graph {
    fn add_directed_edges(&mut self,
                          edges:&ListOfEdges) {
        for (u,v) in edges {
            self.outedges[*u].push(*v);
        }
    }

    fn create_directed(n:usize,edges:&ListOfEdges)
                                            -> Graph {
        let mut g = Graph{n,outedges:vec![vec![];n]};
        g.add_directed_edges(edges);
        g
    }

    fn create_undirected(n:usize,edges:&ListOfEdges)
                                            -> Graph {
        let mut g = Self::create_directed(n,edges);
        g.add_directed_edges(&reverse_edges(edges));
        g
    }
}
```

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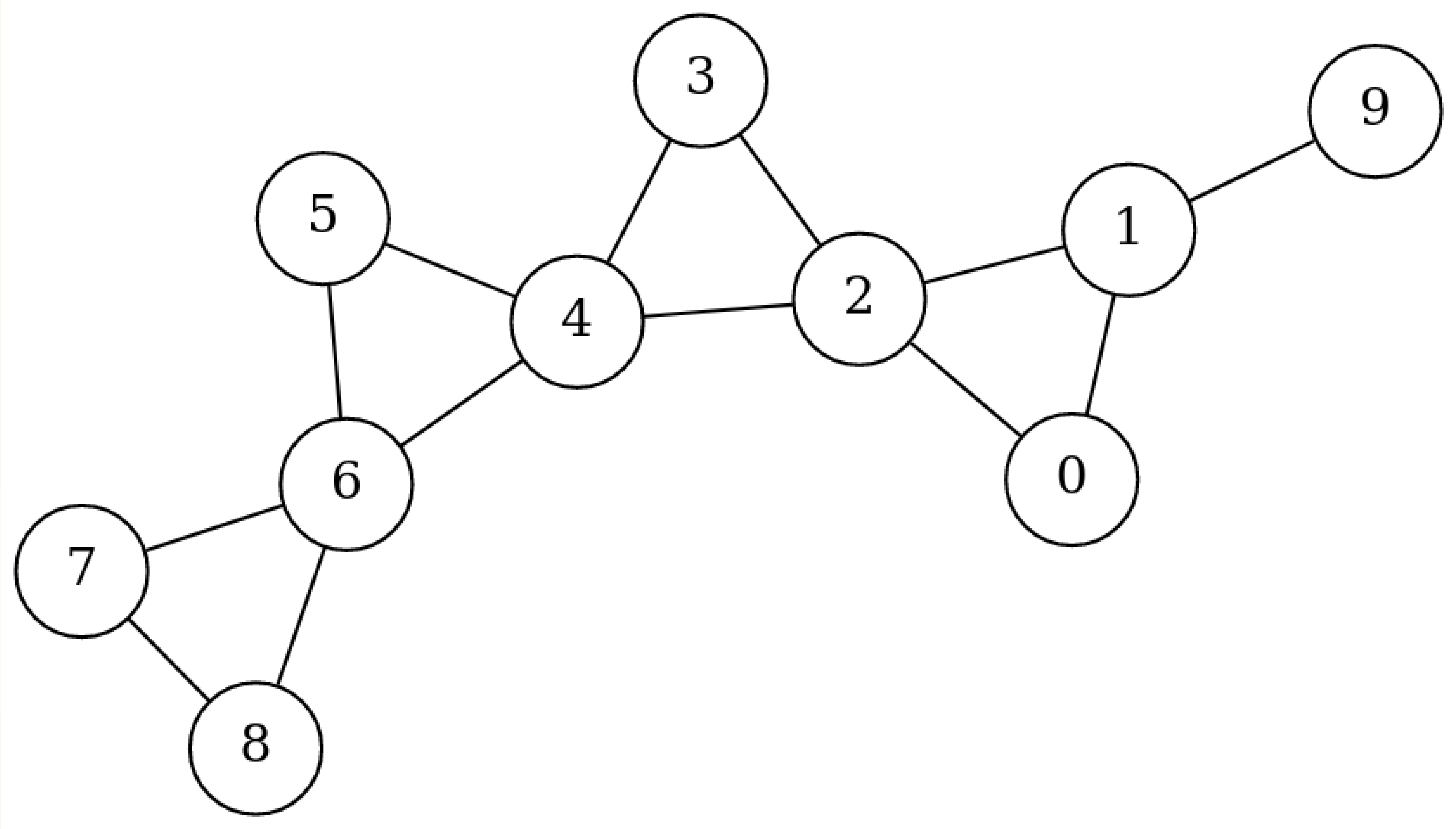
4. BONUS CONTENT: STRONGLY CONNECTED COMPONENTS

SAMPLE GRAPH




```
In [4]: let n: usize = 10;
        let edges: ListOfEdges = vec![(0,1),(0,2),(1,2),(2,4),(2,3),(4,3),(4,5),(5,6),(4,6),(6,8),(6,7),(8,7),(1,9)];
        let graph = Graph::create_undirected(n,&edges);
        graph

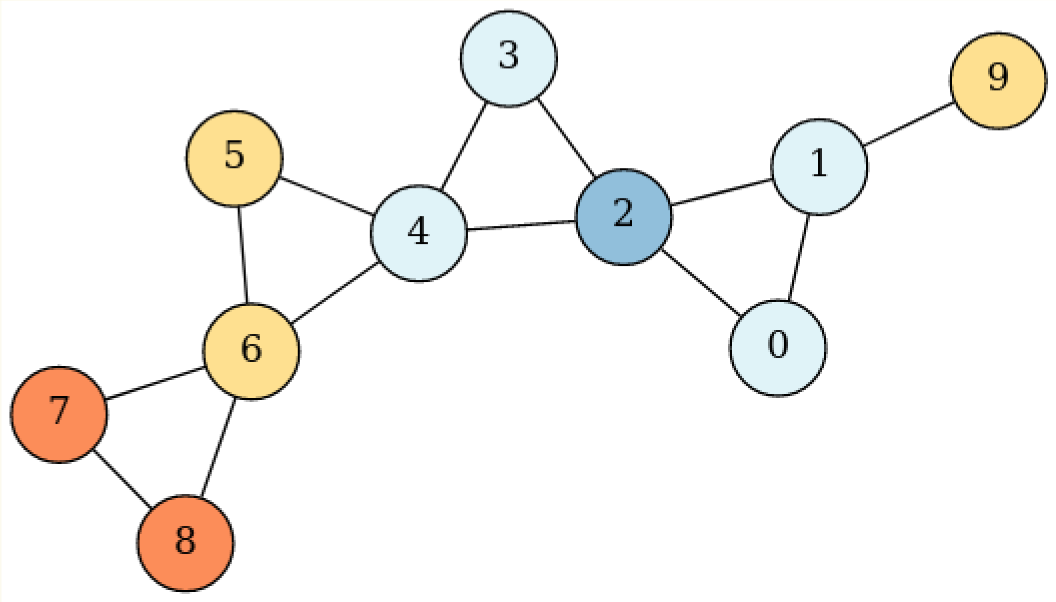
Out[4]: Graph { n: 10, outedges: [[1, 2], [2, 9, 0], [4, 3, 0, 1], [2, 4], [3, 5, 6, 2], [6, 4], [8, 7, 5, 4], [6, 8], [7, 6], [1]] }
```

BREADTH-FIRST SEARCH (BFS)

General idea:

- start from some vertex and explore its neighbors (distance 1)
- then explore neighbors of neighbors (distance 2)
- then explore neighbors of neighbors of neighbors (distance 3)
- ...

Our example: start from vertex 2



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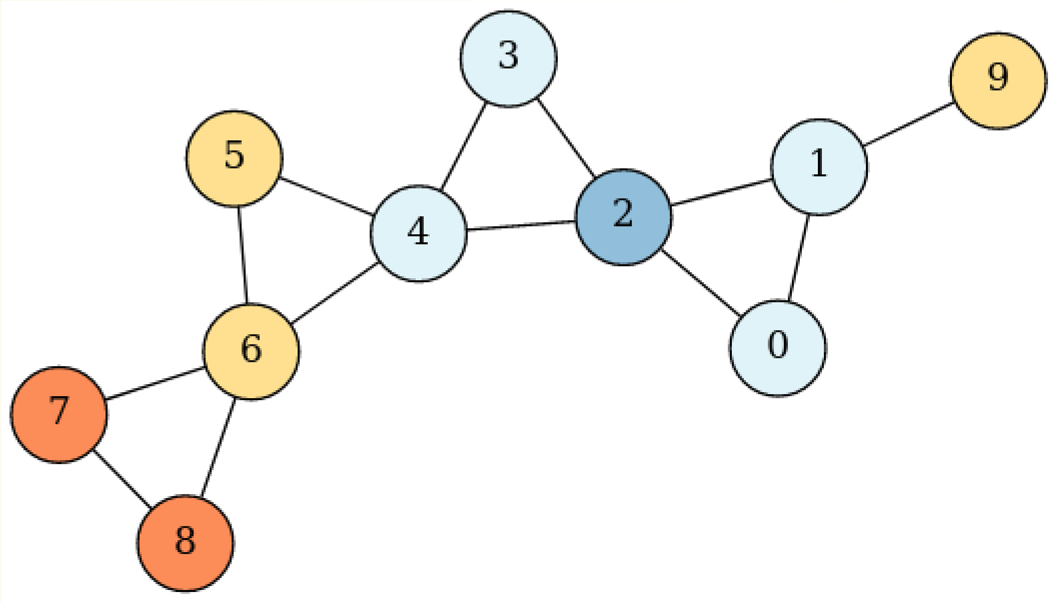

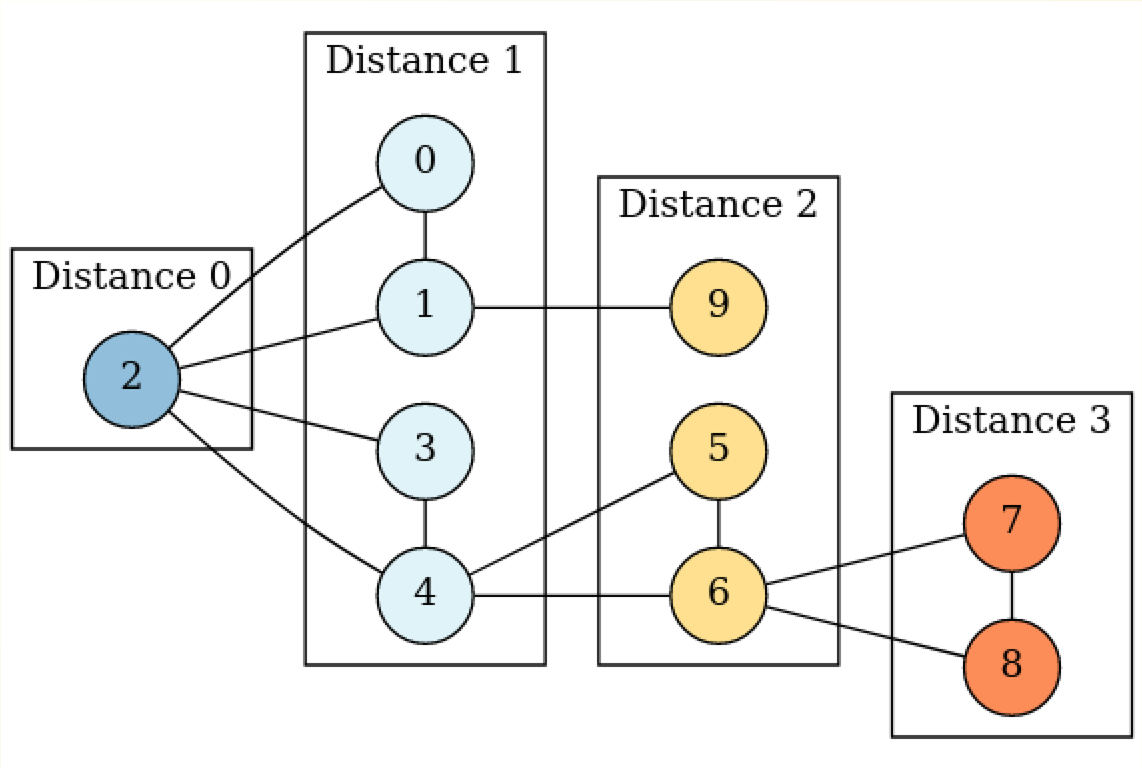

IMPLEMENTATION: COMPUTE DISTANCES FROM VERTEX 2 VIA BFS

`distance[v]`: distance of `v` from vertex 2 (`None` is unknown)

In [5]:
```
let start: Vertex = 2; // <= we'll start from this vertex

let mut distance: Vec<Option<u32>> = vec![None;graph.n];
distance[start] = Some(0); // <= we know this distance
distance
```

Out[5]:
```
[None, None, Some(0), None, None, None, None, None, None, None]
```

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distance
```

Out[5]:
```
[None, None, Some(0), None, None, None, None, None, None, None]
```

`queue`: vertices to consider, they will arrive layer by layer

In [6]:
```
use std::collections::VecDeque;
let mut queue: VecDeque<Vertex> = VecDeque::new();
queue.push_back(start);
queue
```

Out[6]:
```
[2]
```

IMPLEMENTATION: COMPUTE DISTANCES FROM VERTEX 2 VIA BFS

Main loop:

- consider vertices one by one
- add their new neighbors to the processing queue

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In [7]:
```
println!("{:?}",queue);
while let Some(v) = queue.pop_front() { // new unprocessed vertex
    println!("{:?}",queue);
    for u in graph.outedges[v].iter() {
        if let None = distance[*u] { // consider all unprocessed neighbors of v
            distance[*u] = Some(distance[v].unwrap() + 1);
            queue.push_back(*u);
            println!("{:?}",queue);
        }
    }
};
```

```
[2]
[]
[4]
[4, 3]
[4, 3, 0]
[4, 3, 0, 1]
[3, 0, 1]
[3, 0, 1, 5]
[3, 0, 1, 5, 6]
[0, 1, 5, 6]
[1, 5, 6]
```

IMPLEMENTATION: COMPUTE DISTANCES FROM VERTEX 2 VIA BFS

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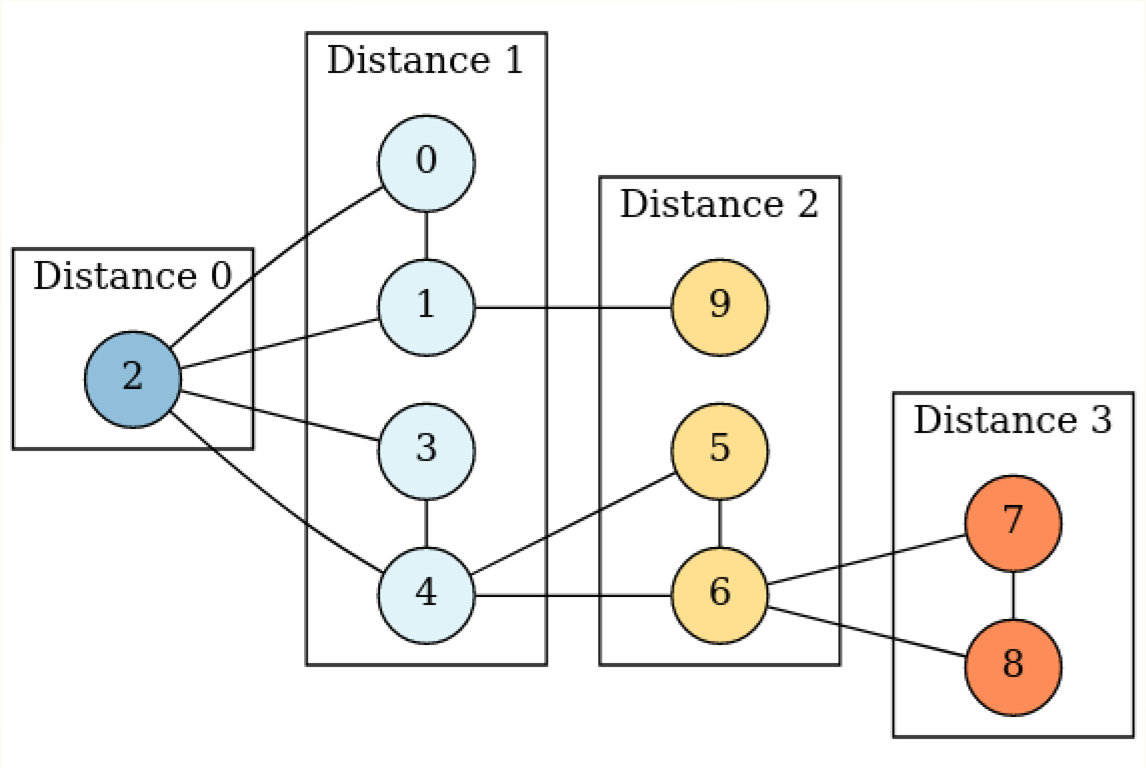
        if let None = distance[*u] { // consider all unprocessed neighbors of v
            distance[*u] = Some(distance[v].unwrap() + 1);
            queue.push_back(*u);
            println!("{:?}",queue);
        }
    }
};
```

```
[2]
[]
[4]
[4, 3]
[4, 3, 0]
[4, 3, 0, 1]
[3, 0, 1]
[3, 0, 1, 5]
[3, 0, 1, 5, 6]
[0, 1, 5, 6]
[1, 5, 6]
[5, 6]
[5, 6, 9]
[6, 9]
[9]
[9, 8]
[9, 8, 7]
[8, 7]
[7]
[]
```

IMPLEMENTATION: COMPUTE DISTANCES FROM VERTEX 2 VIA BFS

Compare results:



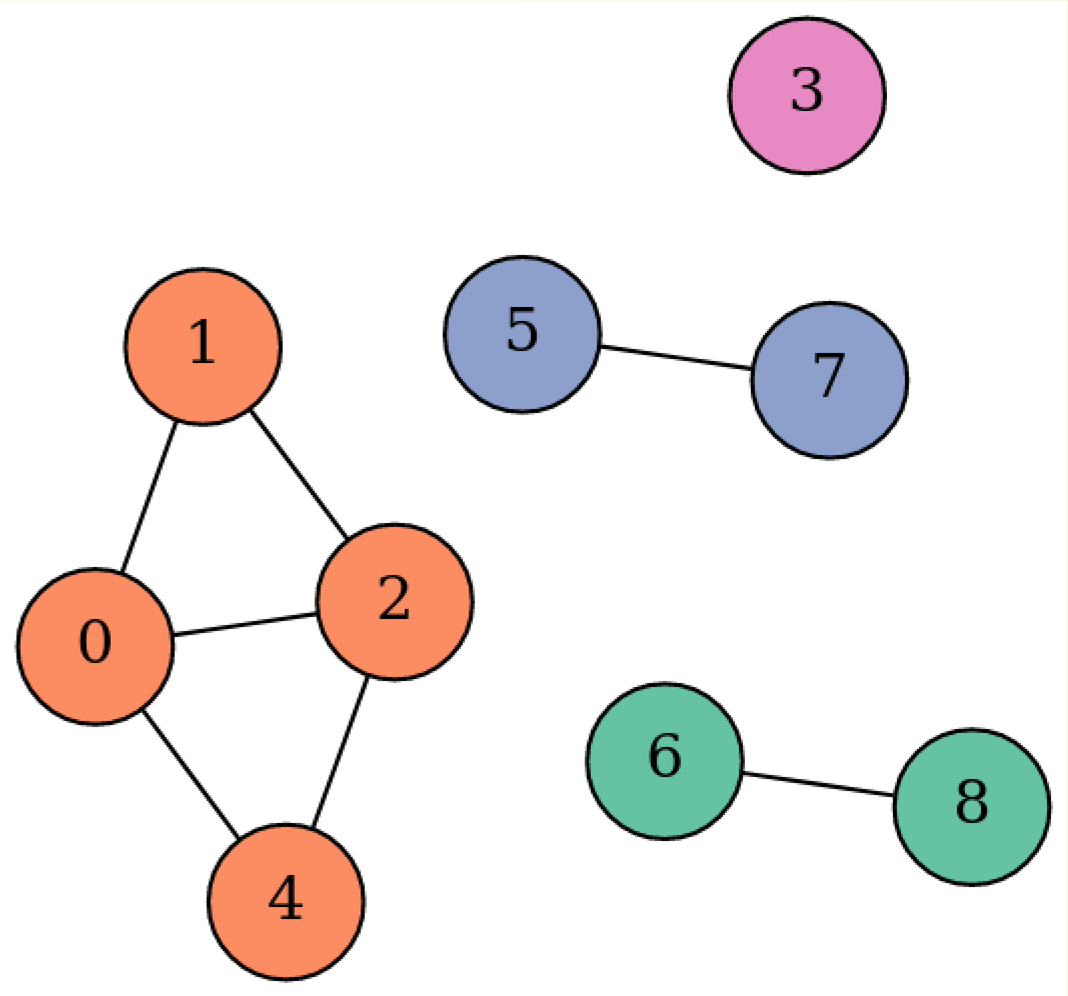
In [8]:
```
print!("vertex:distance");
for v in 0..graph.n {
    print!("   {}:{}",v,distance[v].unwrap());
}
println!();
```

```
vertex:distance   0:1   1:1   2:0   3:1   4:1   5:2   6:2   7:3   8:3   9:2
```

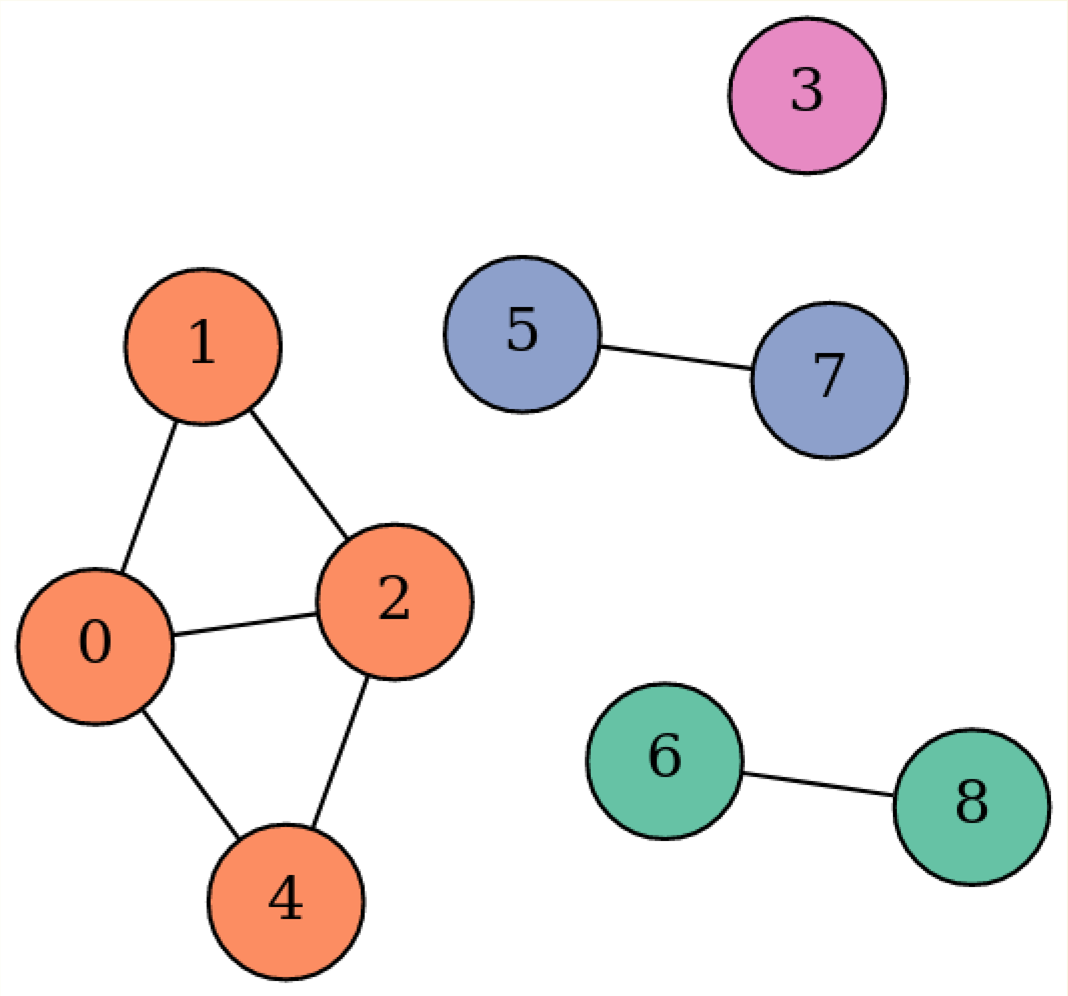
CONNECTED COMPONENTS VIA BFS

Connected component (in an undirected graph):

a maximal set of vertices that are connected



CONNECTED COMPONENTS VIA BFS



Connected component (in an undirected graph):

> a maximal set of vertices that are connected

Sample graph:

In [9]:
```
let n: usize = 9;
let edges: Vec<(Vertex,Vertex)> = vec![(0,1),(0,2),(1,2),(2,4),(0,4),(5,7),(6,8)];
let graph = Graph::create_undirected(n, &edges);
```

DISCOVERING VERTICES OF A CONNECTED COMPONENT VIA BFS

`component[v]`: `v`'s component's number (`None` ≡ not assigned yet)

In [10]:
```
type Component = usize;

fn mark_component_bfs(vertex:Vertex, graph:&Graph, component:&mut Vec<Option<Component>>, component_no:Component) {
    component[vertex] = Some(component_no);

    let mut queue = std::collections::VecDeque::new();
    queue.push_back(vertex);

    while let Some(v) = queue.pop_front() {
        for w in graph.outedges[v].iter() {
            if let None = component[*w] {
                component[*w] = Some(component_no);
                queue.push_back(*w);
            }
        }
    }
}
```

MARKING ALL CONNECTED COMPONENTS

Loop over all unassigned vertices and assign component numbers

In [11]:
```
let mut component: Vec<Option<Component>> = vec![None;n];
let mut component_count = 0;
for v in 0..n {
    if let None = component[v] {
        component_count += 1;
        mark_component_bfs(v, &graph, &mut component, component_count);
    }
};
```

MARKING ALL CONNECTED COMPONENTS

Loop over all unassigned vertices and assign component numbers

In [11]:

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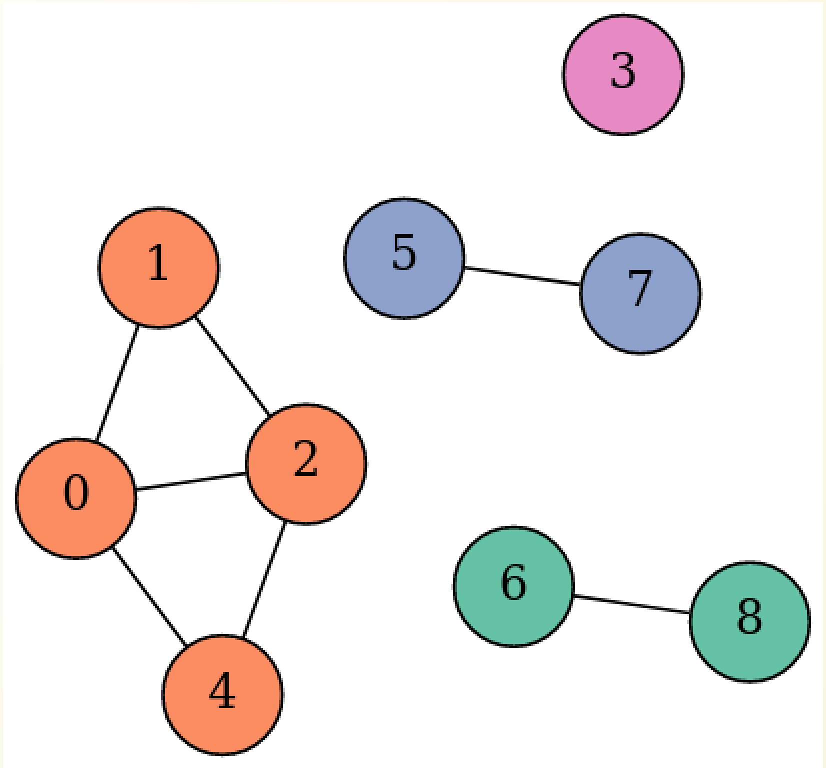
let mut component: Vec<Option<Component>> = vec![None;n];
let mut component_count = 0;
for v in 0..n {
    if let None = component[v] {
        component_count += 1;
        mark_component_bfs(v, &graph, &mut component, component_count);
    }
};
```

In [12]:

```
// Let's verify the assignment!
print!("{} components:\n[  ",component_count);
for v in 0..n {
    print!("{}:{}  ",v,component[v].unwrap());
}
println!("]\n");
```

```
4 components:
[  0:1  1:1  2:1  3:2  4:1  5:3  6:4  7:3  8:4  ]
```



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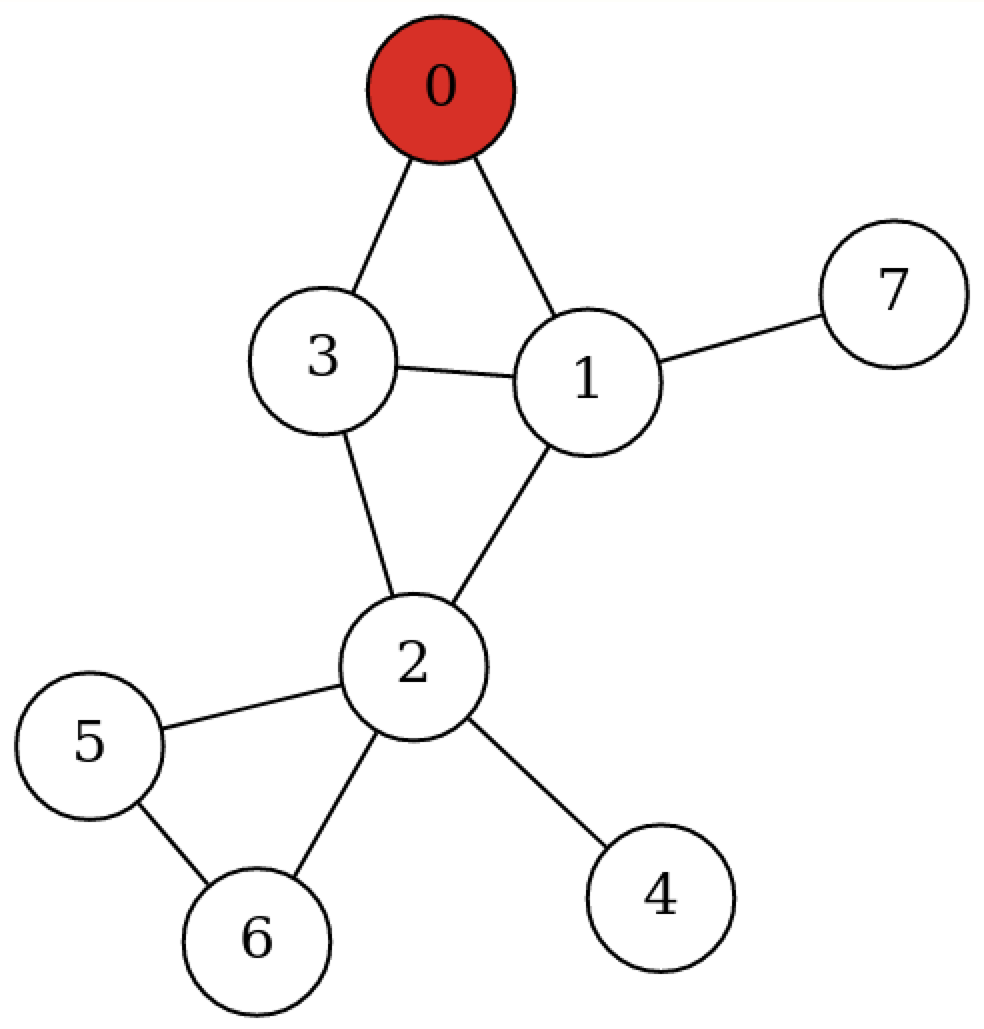
DEPTH–FIRST SEARCH (DFS)

General idea:

- keep going to an unvisited vertex
- when stuck make a step back and try again



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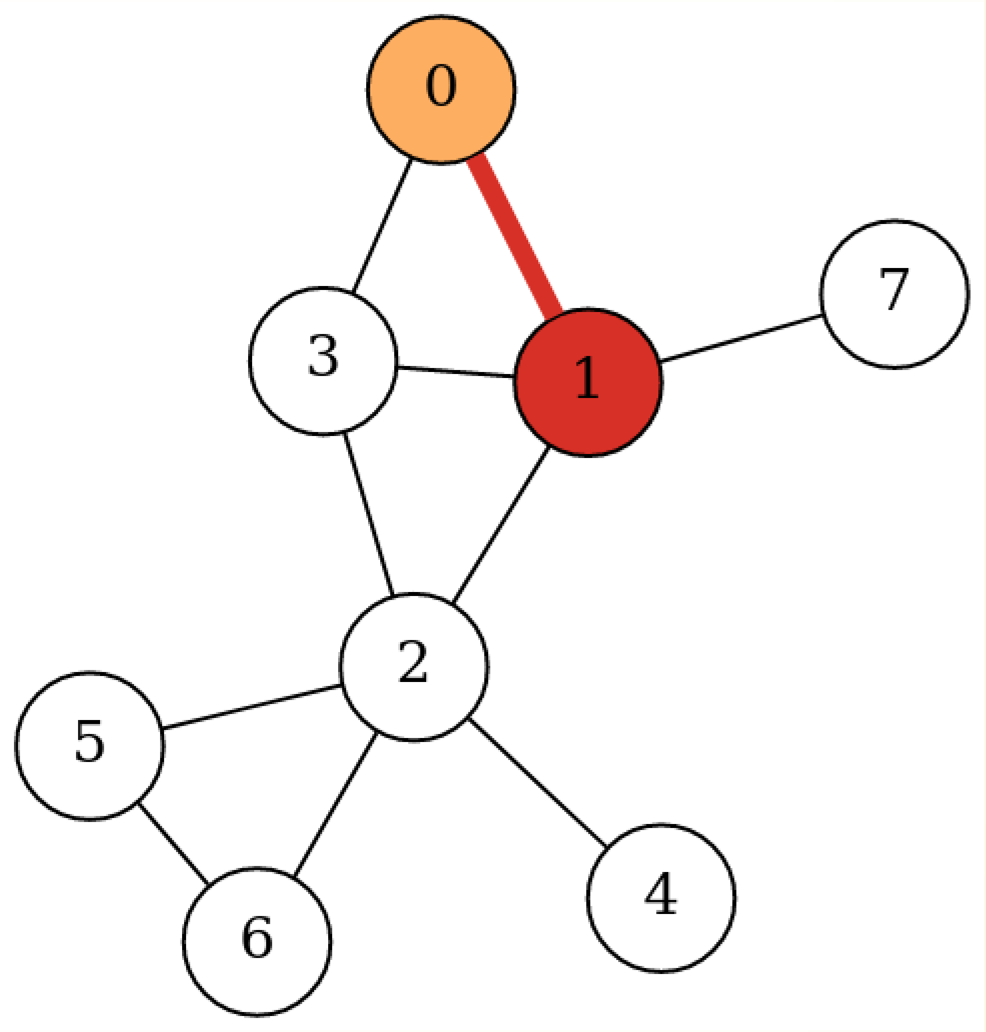

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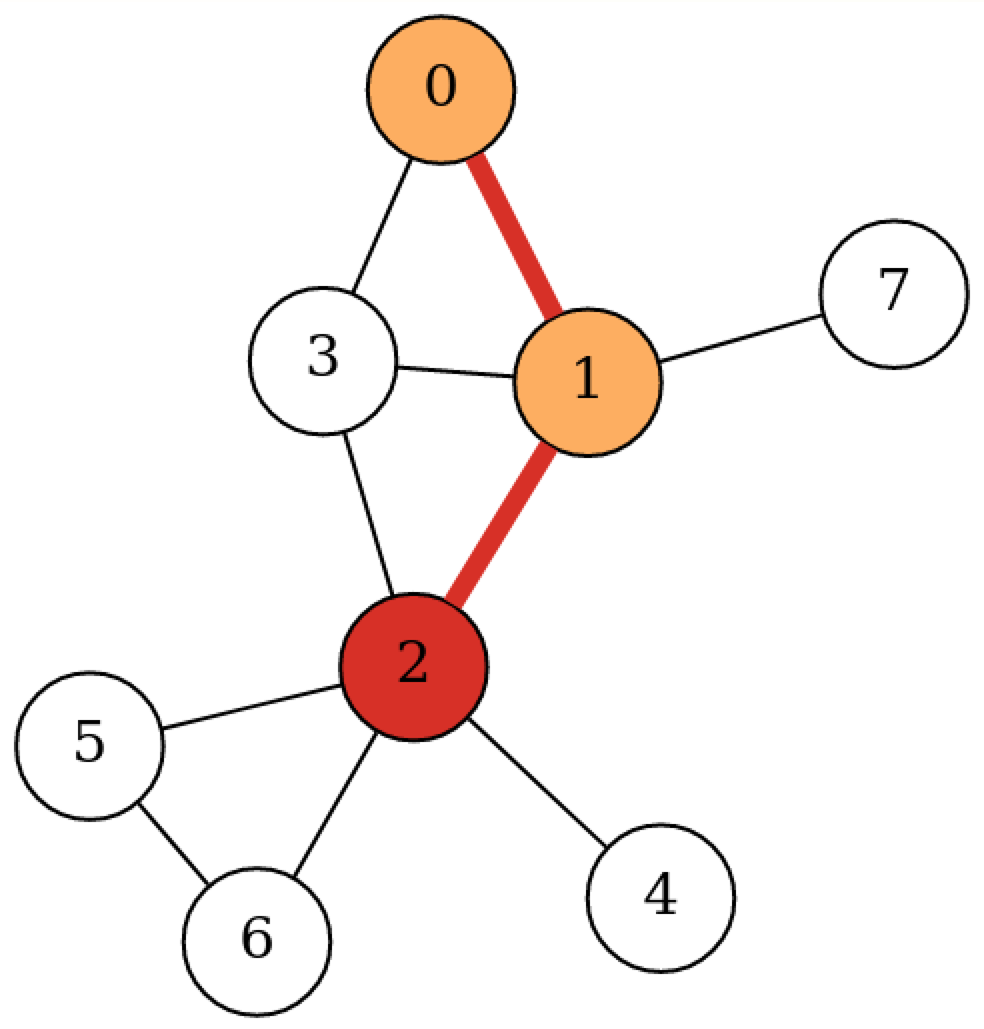

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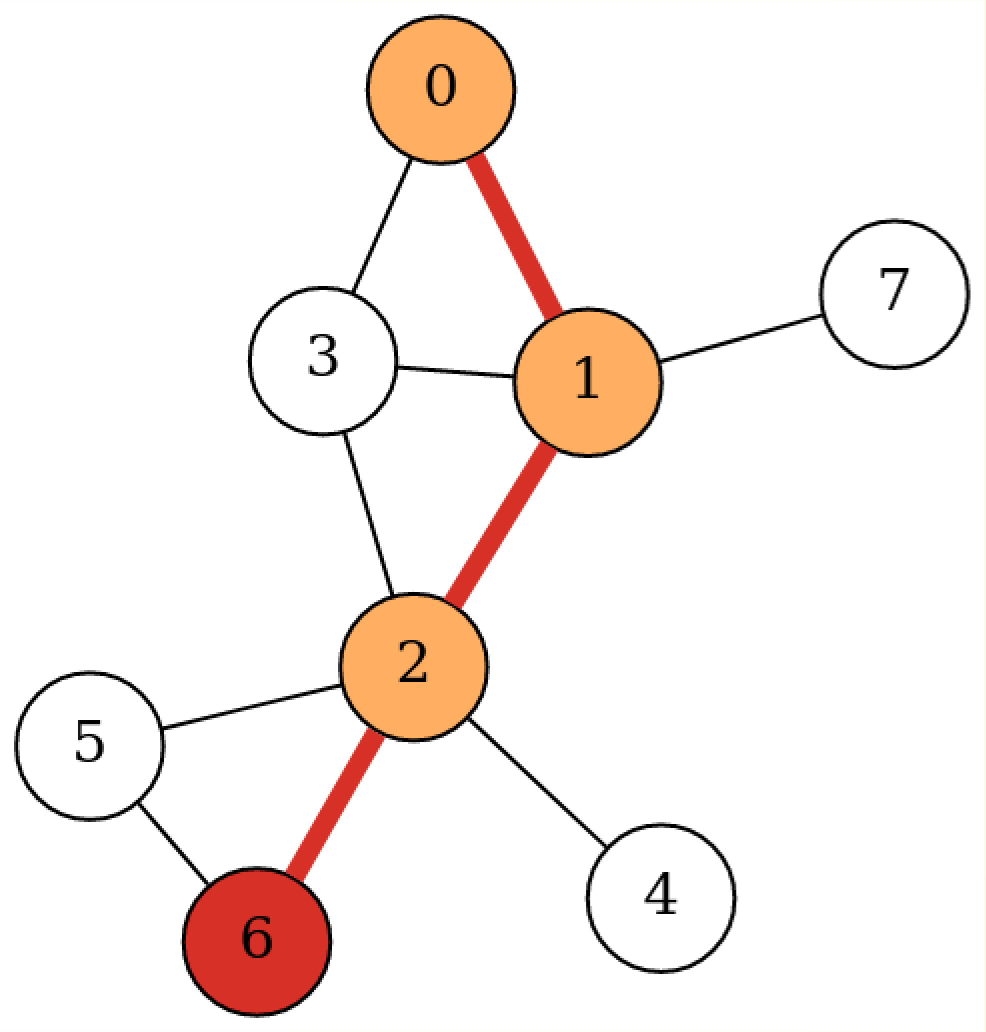
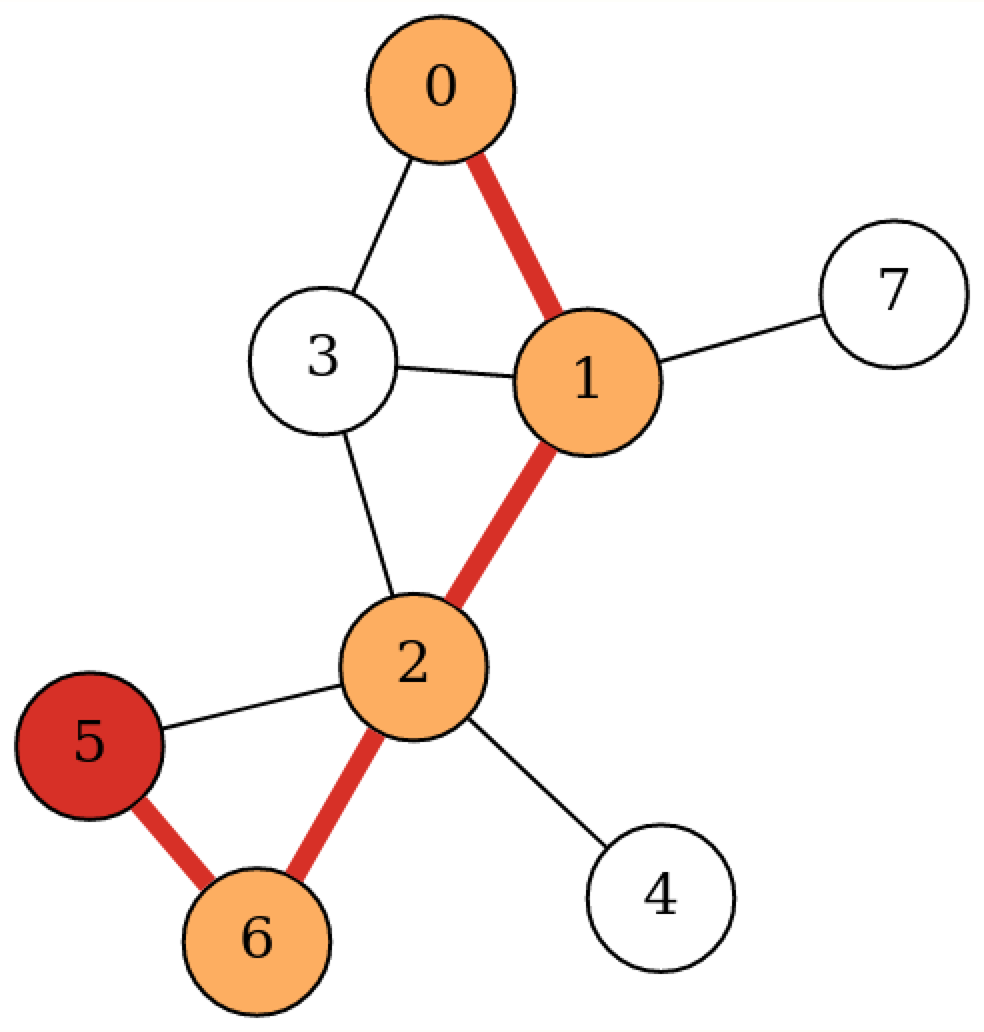


DEPTH–FIRST SEARCH (DFS)

General idea:

- keep moving to an unvisited neighbor
- when stuck make a step back and try again



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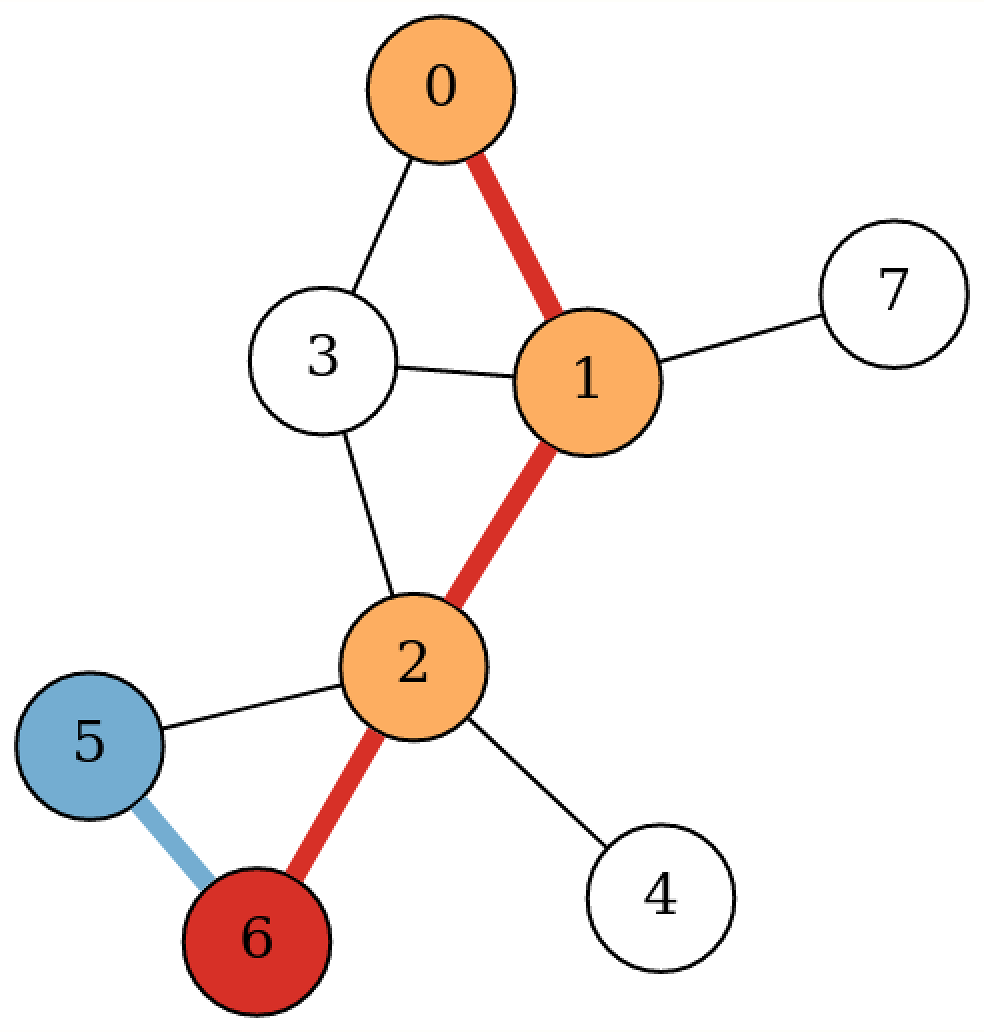

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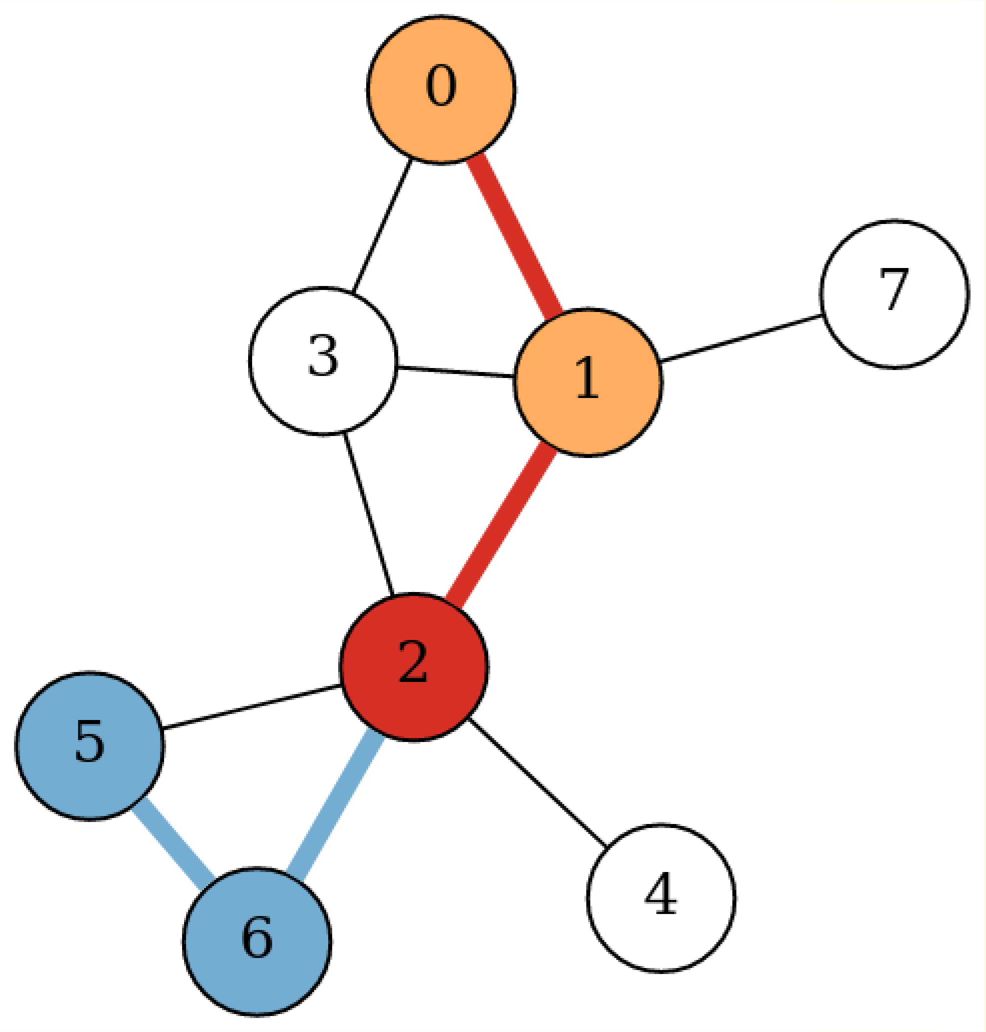

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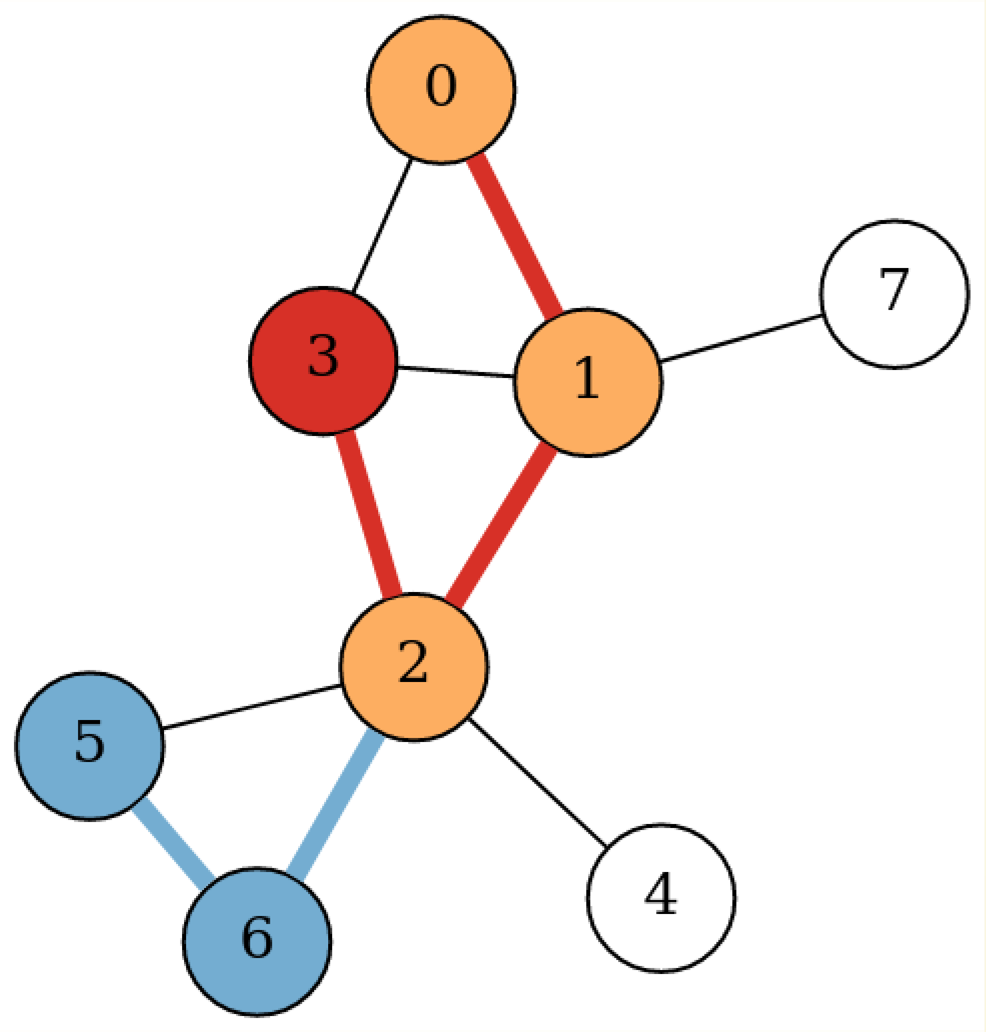

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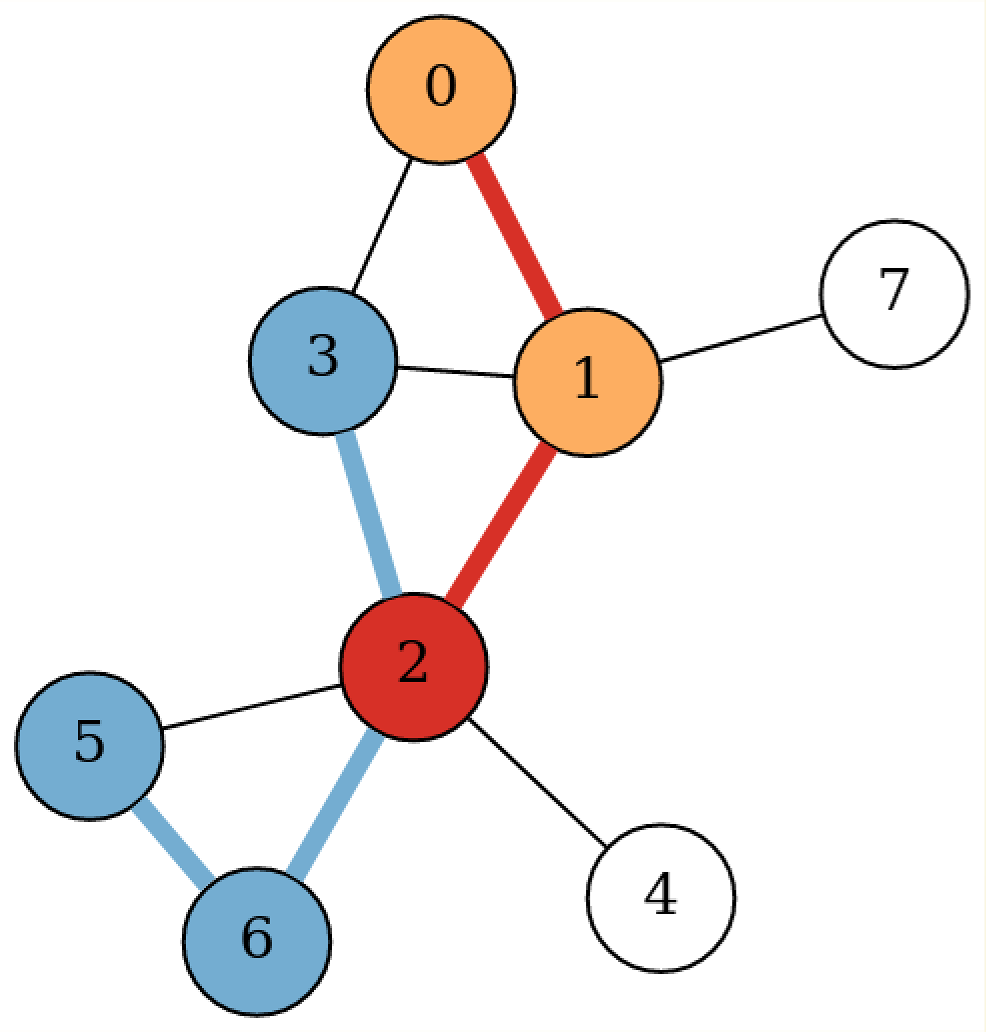

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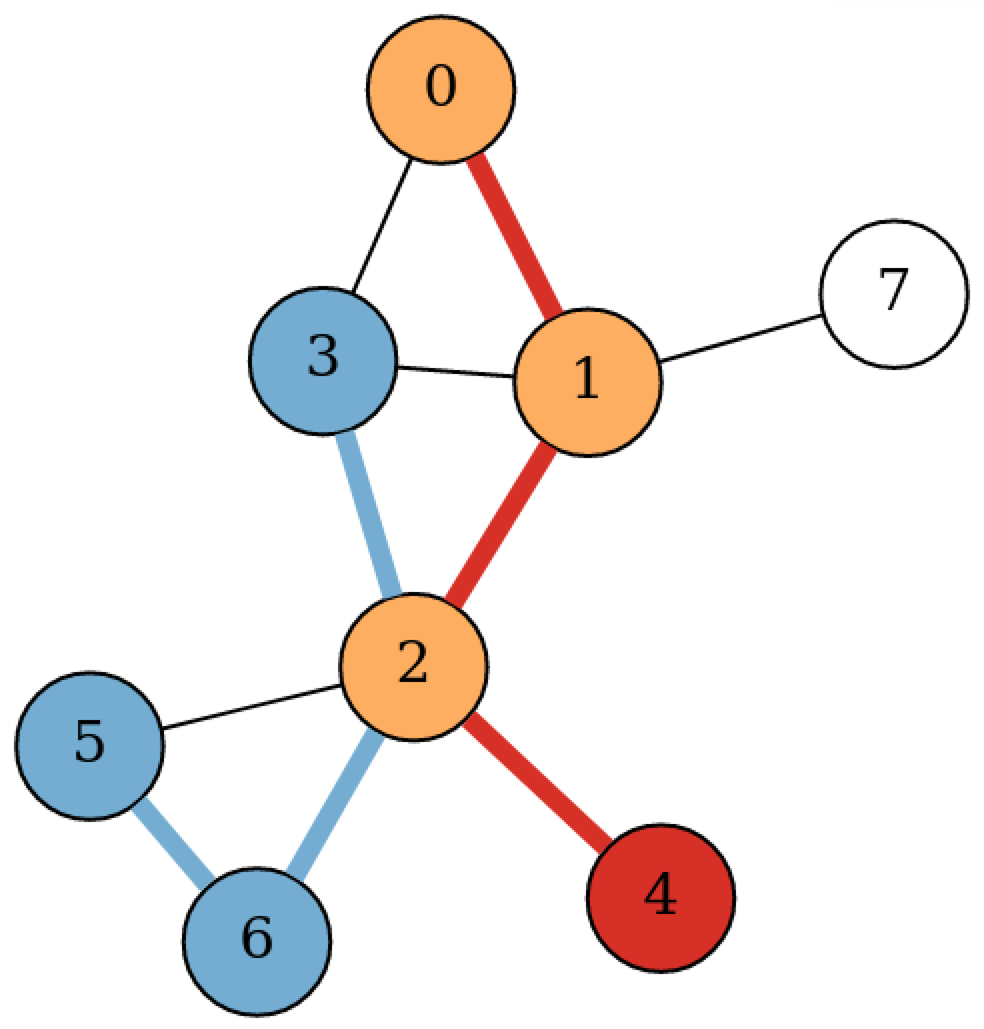

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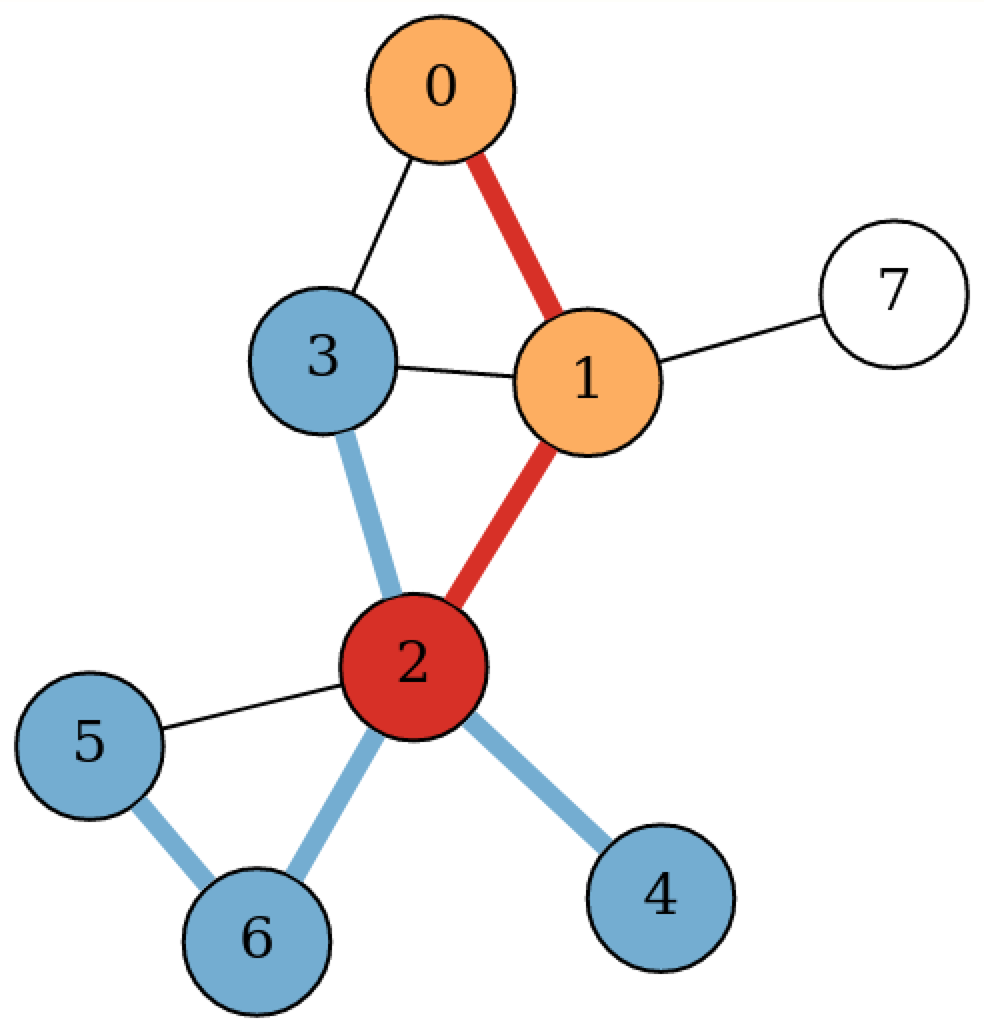

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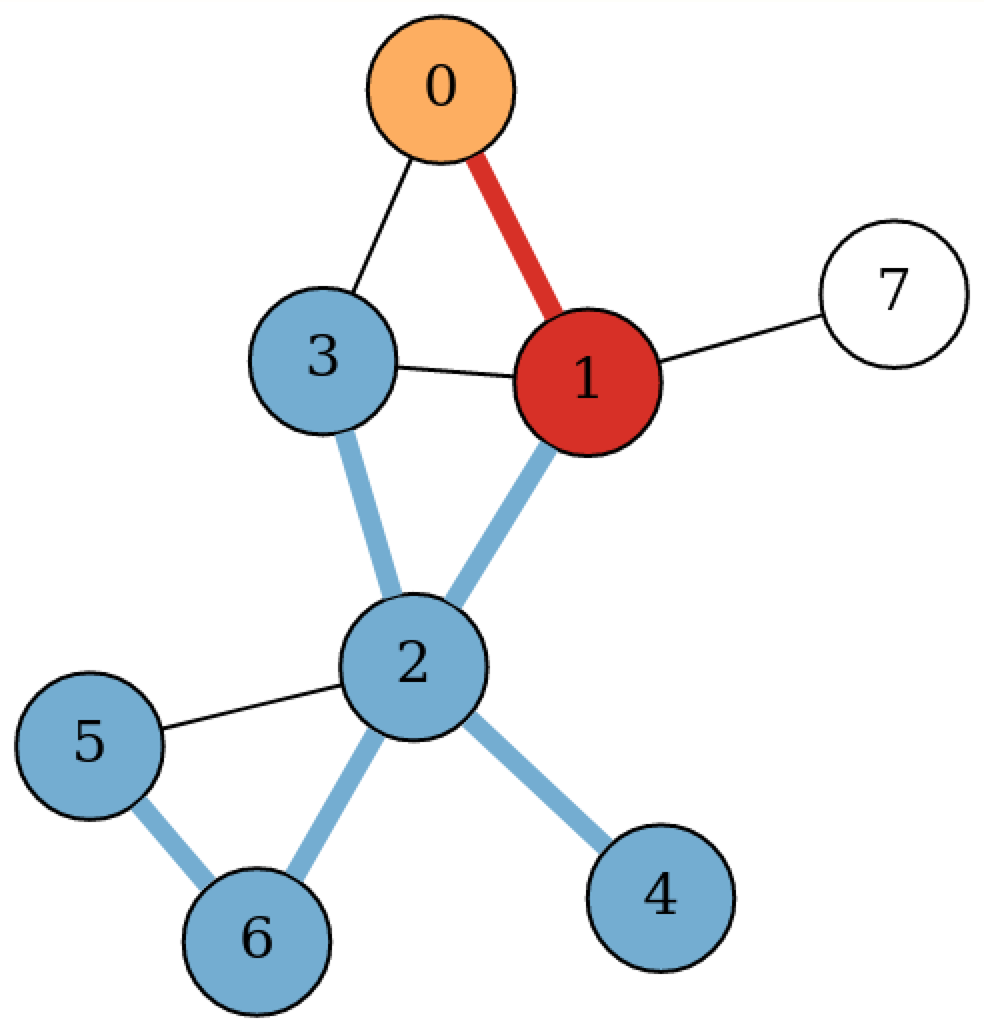

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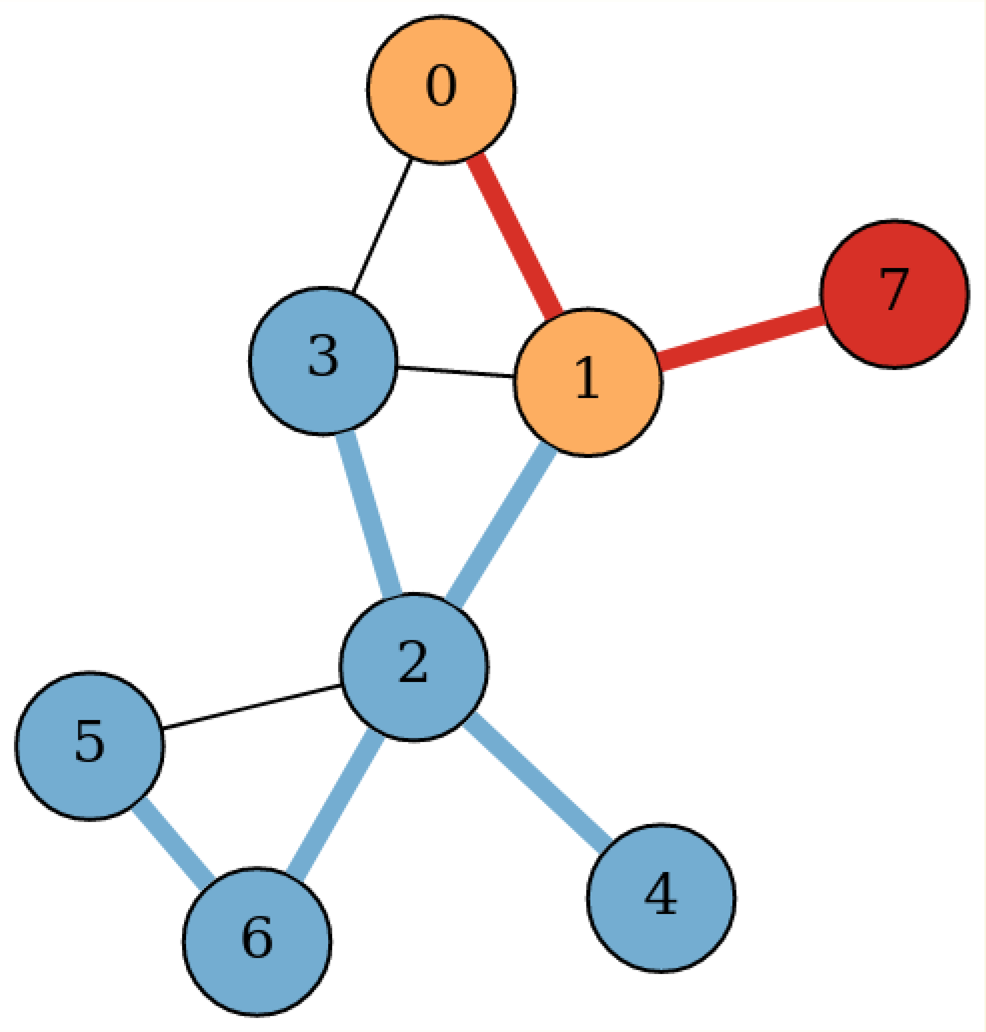

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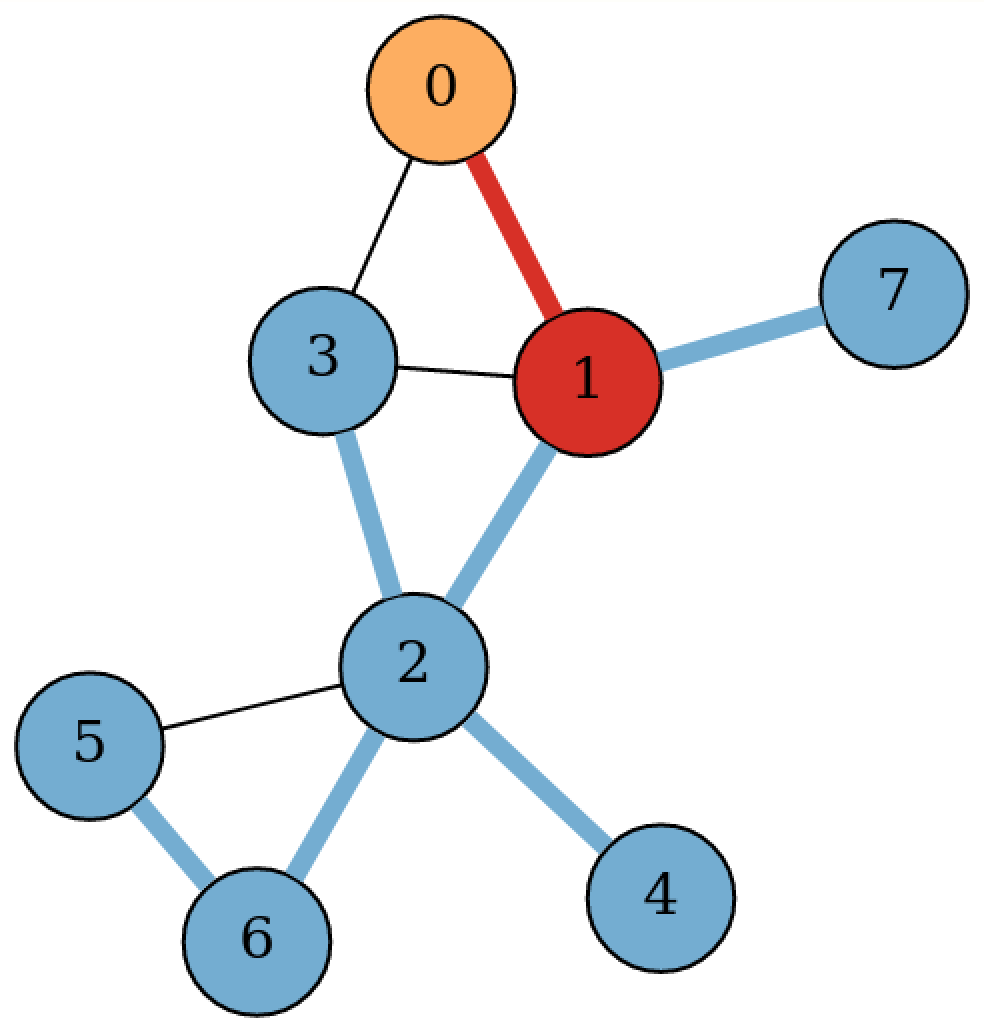

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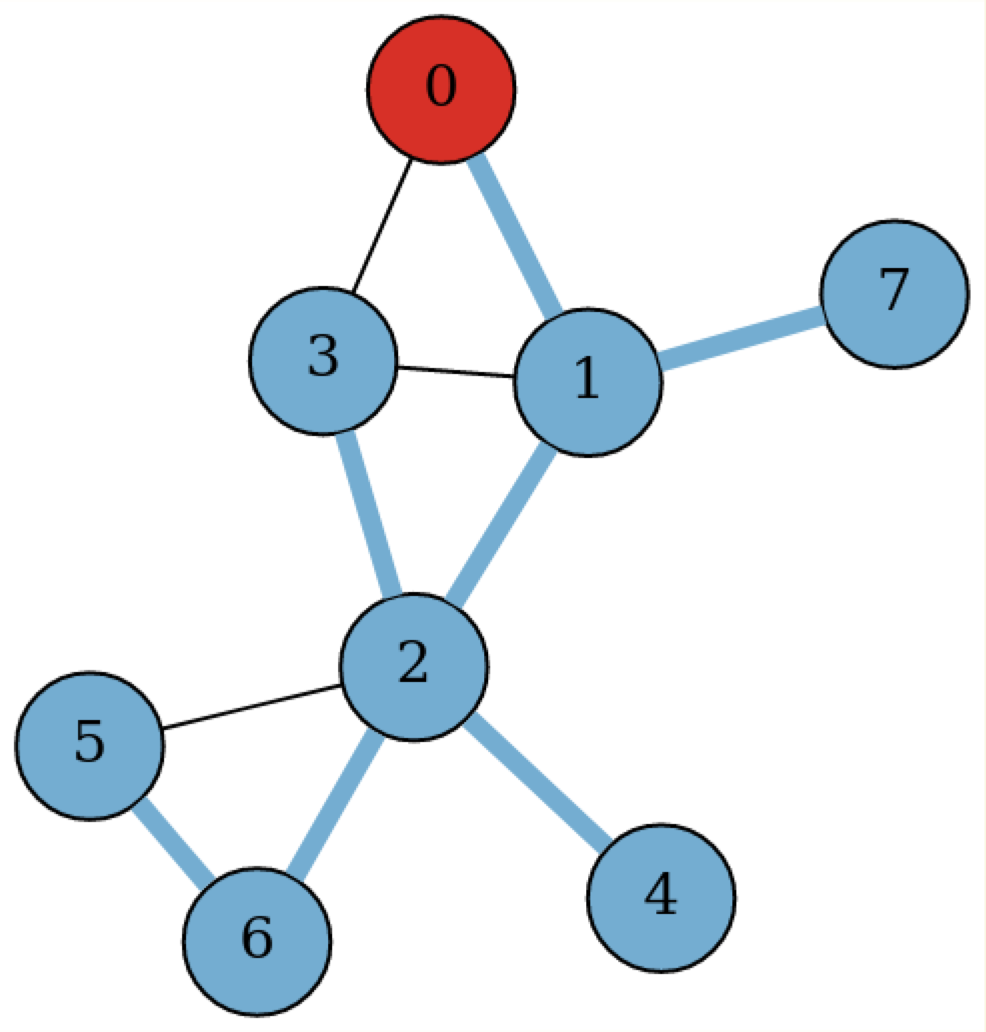

CONNECTED COMPONENTS VIA DFS

Recursive DFS exploration:

In [13]:
```
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    component[vertex] = Some(component_no);
    for w in graph.outedges[vertex].iter() {
        if let None = component[*w] {
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        }
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        }
    }
}
```

Going over all components and assigning vertices:

In [14]:
```
let mut component = vec![None;graph.n];
let mut component_count = 0;

for v in 0..graph.n {
    if let None = component[v] {
        component_count += 1;
        mark_component_dfs(v,&graph,&mut component,component_count);
    }
};
```

CONNECTED COMPONENTS VIA DFS

Let's verify the results:

In [15]:
```
print!("{} components:\n[   ",component_count);
for v in 0..n {
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}
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```

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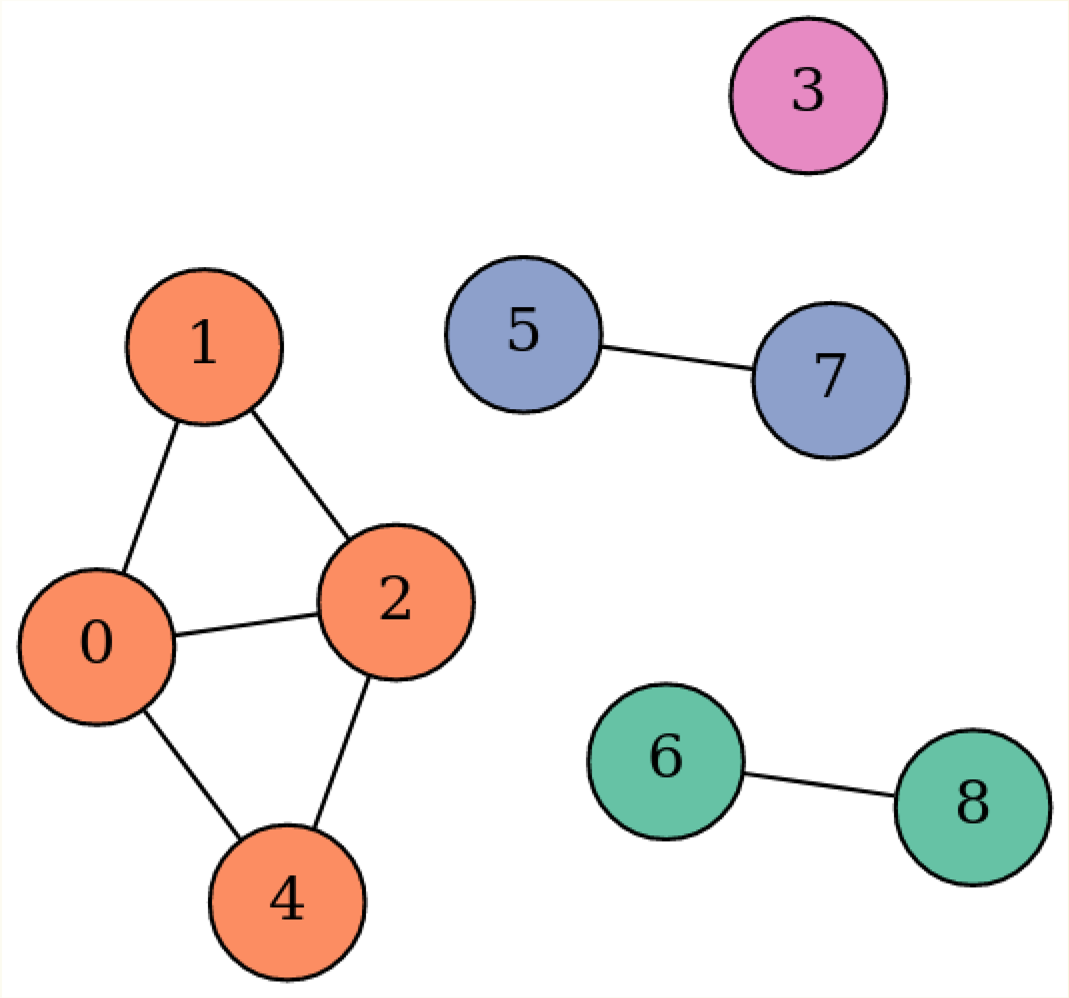
```



BFS VS. DFS

BFS

- gives graph distances between vertices (fundamental problem!)
- connectivity

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DFS

- What is it good for?

BFS VS. DFS

BFS

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- connectivity

DFS

- What is it good for?

LOTS OF THINGS!

Examples:

- find edges/vertices crucial for connectivity
- orient edges of a graph so it is still connected
- strongly connected components in directed graphs

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STRONG CONNECTIVITY

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What if you can get from v to w, but not from w to v?

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Strongly connected component:
a maximal set of vertices such that you can get from any of them to any other one

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Fact: There is a unique decomposition

FIND THE UNIQUE DECOMPOSITION VIA TWO DFS RUNS

GENERAL IDEA

First DFS:

- maintain auxiliary stack S
- visit all vertices, starting DFS multiple times from unvisited vertices as needed
- put each vertex, when done going over its neighbors, on the stack

FIND THE UNIQUE DECOMPOSITION VIA TWO DFS RUNS

GENERAL IDEA

First DFS:

- maintain auxiliary stack S
- visit all vertices, starting DFS multiple times from unvisited vertices as needed
- put each vertex, when done going over its neighbors, on the stack

Second DFS:

- **reverse edges of the graph!!!**
- consider vertices in order from the stack
- for each unvisited vertex, start DFS: it will visit a new strongly connected component

IMPLEMENTATION

In [16]:
```
let n: usize = 7;
let edges: ListOfEdges = vec![(0,1),(1,2),(2,0),(3,4),(4,5),(5,3),(2,3),(6,5)];
let graph = Graph::create_directed(n, &edges);
let graph_reverse = Graph::create_directed(n,&reverse_edges(&edges));
println!("{:?}\n{:?}",graph,graph_reverse);
```

```
Graph { n: 7, outedges: [[1], [2], [0, 3], [4], [5], [3], [5]] }
Graph { n: 7, outedges: [[2], [0], [1], [5, 2], [3], [4, 6], []] }
```

IMPLEMENTATION (FIRST DFS)

In [17]:
```
let mut stack: Vec<Vertex> = Vec::new();
let mut visited = vec![false;graph.n];
```

IMPLEMENTATION (FIRST DFS)

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```

In [18]:
```
fn dfs_collect_stack(v:Vertex, graph:&Graph, stack:&mut Vec<Vertex>, visited:&mut Vec<bool>) {
    if !visited[v] {
        visited[v] = true;
        for w in graph.outedges[v].iter() {
            dfs_collect_stack(*w, graph, stack, visited);
        }
        stack.push(v);
    }
}
```

IMPLEMENTATION (FIRST DFS)

In [17]:
```
let mut stack: Vec<Vertex> = Vec::new();
let mut visited = vec![false;graph.n];
```

In [18]:
```
fn dfs_collect_stack(v:Vertex, graph:&Graph, stack:&mut Vec<Vertex>, visited:&mut Vec<bool>) {
    if !visited[v] {
        visited[v] = true;
        for w in graph.outedges[v].iter() {
            dfs_collect_stack(*w, graph, stack, visited);
        }
        stack.push(v);
    }
}
```

In [19]:
```
for v in 0..graph.n {
    dfs_collect_stack(v,&graph,&mut stack,&mut visited);
};
stack
```

Out[19]: [5, 4, 3, 2, 1, 0, 6]

IMPLEMENTATION (SECOND DFS, REVERSED GRAPH)

In [20]:
```
let mut component: Vec<Option<Component>> = vec![None;graph.n];
let mut component_count = 0;

while let Some(v) = stack.pop() {
    if let None = component[v] {
        component_count += 1;
        mark_component_dfs(v, &graph_reverse, &mut component, component_count);
    }
};
```

IMPLEMENTATION (SECOND DFS, REVERSED GRAPH)

In [20]:
```
let mut component: Vec<Option<Component>> = vec![None;graph.n];
let mut component_count = 0;

while let Some(v) = stack.pop() {
    if let None = component[v] {
        component_count += 1;
        mark_component_dfs(v, &graph_reverse, &mut component, component_count);
    }
};
```

In [21]:
```
print!("{} components:\n[  ",component_count);
for v in 0..n {
    print!("{}:{}  ",v,component[v].unwrap());
}
println!("]\n");
```

```
3 components:
[  0:2  1:2  2:2  3:3  4:3  5:3  6:1  ]
```