Tutorial: Sparse Recovery Using Sparse Matrices

Piotr Indyk

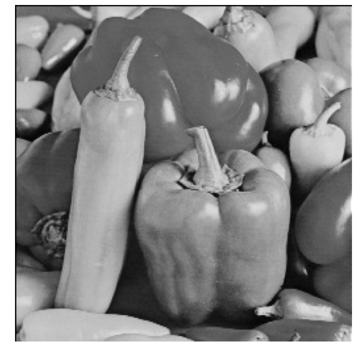
MIT

Problem Formulation

(approximation theory, learning Fourier coeffs, linear sketching, finite rate of innovation, **compressed sensing...)**

- Setup:
  - Data/signal in n-dimensional space : x

    E.g., x is an 256x256 image $\Rightarrow$ n=65536
  - Goal: compress x into a "sketch" Ax ,
    where A is a $m \times n$ "sketch matrix", $m << n$



- Requirements:
  - Plan A: want to recover x from Ax
    - Impossible: underdetermined system of equations
  - Plan B: want to recover an "approximation" x^* of x
    - Sparsity parameter k
    - Informally: want to recover largest k coordinates of x
    - Formally: want x^* such that

      $$\|x^*-x\|_p \le C(k) \min_{x'} \|x'-x\|_q$$

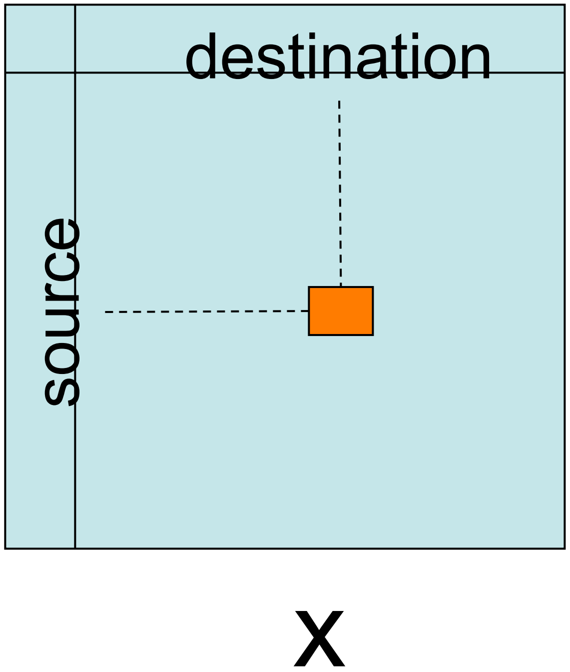
      over all x' that are k-sparse (at most k non-zero entries)

$$\Big[\quad A \quad\Big] \Big[\, x \,\Big] = \Big[\, Ax \,\Big]$$

- Want:
  - Good compression (small $m=m(k,n)$)
  - Efficient algorithms for encoding and recovery
- Why linear compression ?

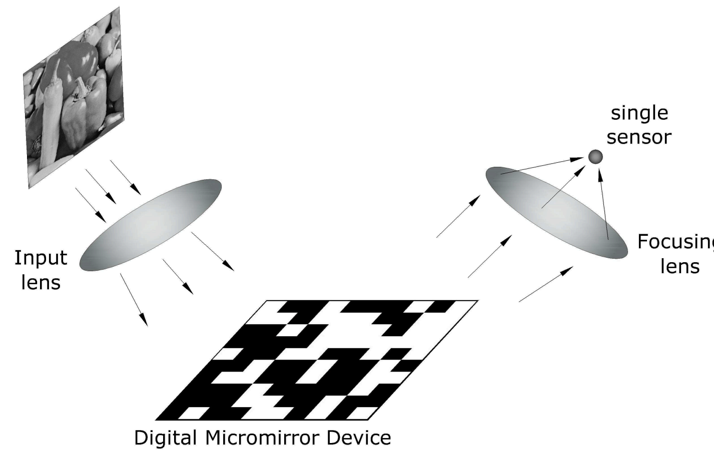
Application I: Monitoring Network Traffic Data Streams

- Router routs packets
  - Where do they come from ?
  - Where do they go to ?
- Ideally, would like to maintain a traffic matrix $x[.,.]$
  - Easy to update: given a (src,dst) packet, increment $x_{src,dst}$
  - Requires way too much space! (2^{32} x 2^{32} entries)
  - Need to compress x, increment easily
- Using linear compression we can:
  - Maintain sketch Ax under increments to x, since
    $$A(x+\Delta) = Ax + A\Delta$$
  - Recover x^* from Ax



Applications, ctd.

- Single pixel camera

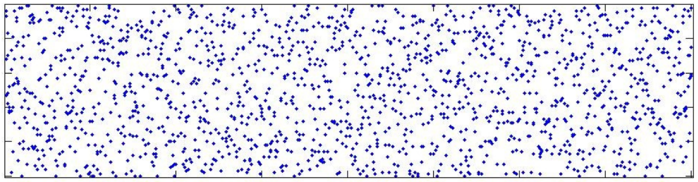


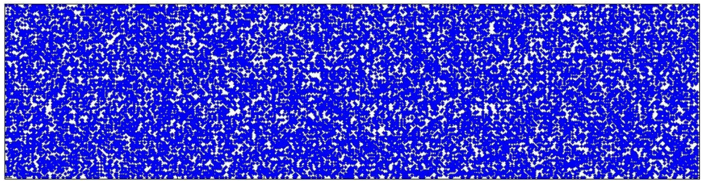
[Wakin, Laska, Duarte, Baron, Sarvotham, Takhar, Kelly, Baraniuk'06]

- Pooling Experiments
  [Kainkaryam, Woolf'08], [Hassibi et al'07], [Dai-Sheikh, Milenkovic, Baraniuk], [Shental-Amir-Zuk'09],[Erlich-Shental-Amir-Zuk'09]

Constructing matrix A

- "Most" matrices A work
  - Sparse matrices:
    - Data stream algorithms
    - Coding theory (LDPCs)
  - Dense matrices:
    - Compressed sensing
    - Complexity/learning theory (Fourier matrices)





- "Traditional" tradeoffs:
  - Sparse: computationally more efficient, explicit
  - Dense: shorter sketches
- Recent results: the "best of both worlds"

Prior and New Results

Paper	Rand. / Det.	Sketch length	Encode time	Column sparsity	Recovery time	Approx

Scale: Excellent | Very Good | Good | Fair

Results

Paper	R/D	Sketch length	Encode time	Column sparsity	Recovery time	Approx
[CCF'02], [CM'06]	R	k log n	n log n	log n	n log n	l2 / l2
	R	k $\log^c$ n	n $\log^c$ n	$\log^c$ n	k $\log^c$ n	l2 / l2
[CM'04]	R	k log n	n log n	log n	n log n	l1 / l1
	R	k $\log^c$ n	n $\log^c$ n	$\log^c$ n	k $\log^c$ n	l1 / l1
[CRT'04] [RV'05]	D	k log(n/k)	nk log(n/k)	k log(n/k)	n^c	l2 / l1
	D	k $\log^c$ n	n log n	k $\log^c$ n	n^c	l2 / l1
[GSTV'06] [GSTV'07]	D	k $\log^c$ n	n $\log^c$ n	$\log^c$ n	k $\log^c$ n	l1 / l1
	D	k $\log^c$ n	n $\log^c$ n	k $\log^c$ n	$k^2 \log^c$ n	l2 / l1
[BGIKS'08]	D	k log(n/k)	n log(n/k)	log(n/k)	n^c	l1 / l1
[GLR'08]	D	k $\log n^{\log\log\log n}$	kn^{1-a}	n^{1-a}	n^c	l2 / l1
[NV'07], [DM'08], [NT'08], [BD'08], [GK'09], …	D	k log(n/k)	nk log(n/k)	k log(n/k)	nk log(n/k) * log	l2 / l1
	D	k $\log^c$ n	n log n	k $\log^c$ n	n log n * log	l2 / l1
[IR'08], [BIR'08],[BI'09]	D	k log(n/k)	n log(n/k)	log(n/k)	n log(n/k)* log	l1 / l1
[GLSP'09]	R	k log(n/k)	n $\log^c$ n	$\log^c$ n	k $\log^c$ n	l2 / l1

Caveats: (1) most "dominated" results not shown (2) only results for general vectors x are displayed
(3) sometimes the matrix type matters (Fourier, etc)

Part I

Paper	R/ D	Sketch length	Encode time	Column sparsity	Recovery time	Approx
[CM'04]	R	k log n	n log n	log n	n log n	l1 / l1

Theorem: There is a distribution over mxn matrices A, m=O(k logn), such that for any x, given Ax, we can recover x* such that

$$||x-x^*||_1 \leq C \, Err_1 \text{ , where } Err_1 = \min_{k\text{-sparse } x'} ||x-x'||_1$$

with probability 1-1/n.

The recovery algorithm runs in O(n log n) time.

This talk:

- Assume $x \geq 0$ – this simplifies the algorithm and analysis; see the original paper for the general case
- Prove the following l_∞/l_1 guarantee: $||x-x^*||_\infty \leq C \, Err_1 /k$

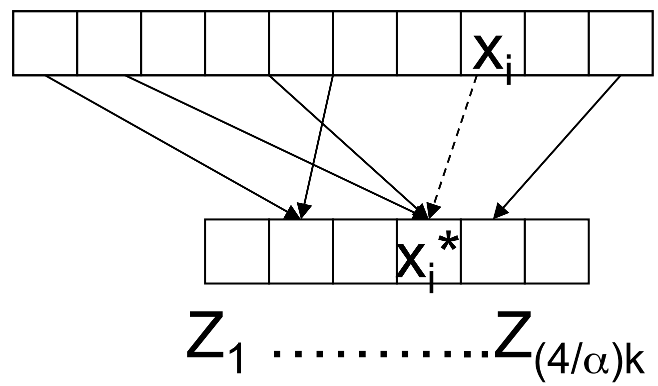
  This is actually stronger than the l_1/l_1 guarantee (cf. [CM'06], see also the Appendix)

Note: [CM'04] originally proved a weaker statement where $||x-x^*||_\infty \leq C||x||_1 /k$. The stronger guarantee follows from the analysis of [CCF'02] (cf. [GGIKMS'02]) who applied it to Err_2

First attempt

- Matrix view:
  - A 0-1 wxn matrix A, with one 1 per column
  - The i-th column has 1 at position h(i), where h(i) be chosen uniformly at random from {1…w}
- Hashing view:
  - Z=Ax
  - h hashes coordinates into "buckets" $Z_1 \ldots Z_w$
- Estimator: $x_i^* = Z_{h(i)}$

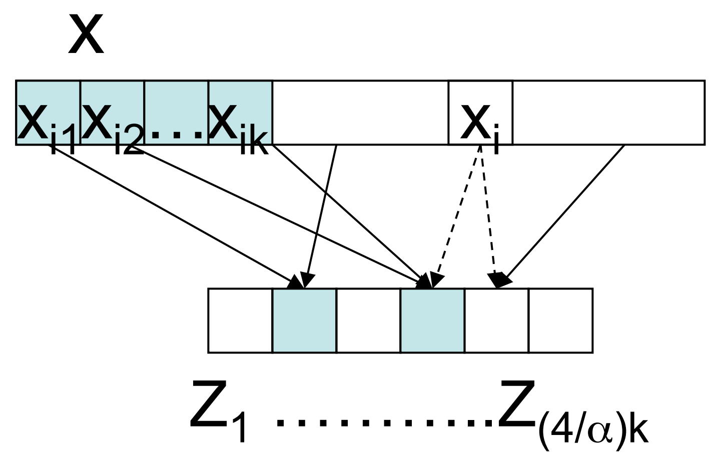
$$\begin{matrix} 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{matrix}$$



Closely related: [Estan-Varghese'03], "counting" Bloom filters

Analysis

- We show
  $$x_i^* \le x_i \pm \alpha \text{ Err}/k$$
  with probability >1/2
- Assume
  $$|x_{i1}| \ge \ldots \ge |x_{im}|$$
  and let $S=\{i1\ldots ik\}$ ("elephants")
- When is $x_i^* > x_i \pm \alpha \text{ Err}/k$?
  - **Event 1**: S and i collide, i.e., $h(i)$ in $h(S-\{i\})$
    Probability: at most $k/(4/\alpha)k = \alpha/4 < 1/4$ (if $\alpha<1$)
  - **Event 2**: many "mice" collide with i., i.e.,
    $$\sum_{t \text{ not in } S \text{ u } \{i\},\, h(i)=h(t)} x_t > \alpha \text{ Err}/k$$
    Probability: at most ¼ (Markov inequality)
- Total probability of "bad" events <1/2



Second try

- Algorithm:
  - Maintain d functions $h_1 \ldots h_d$ and vectors $Z^1 \ldots Z^d$
  - Estimator:

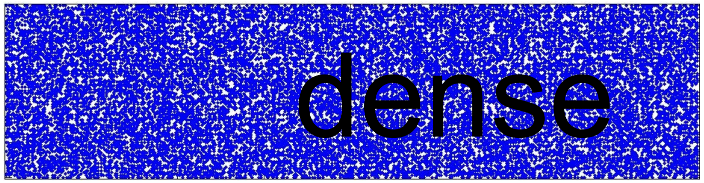
$$X_i^* = \min_t Z^t_{ht(i)}$$

- Analysis:
  - $\Pr[|x_i^* - x_i| \geq \alpha \, Err/k] \leq 1/2^d$
  - Setting $d=O(\log n)$ (and thus $m=O(k \log n)$) ensures that w.h.p

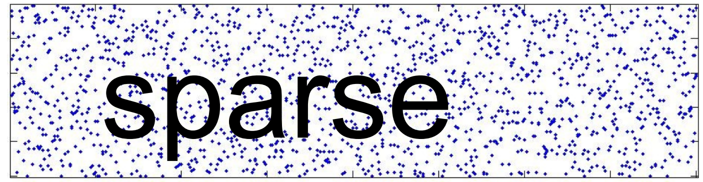
$$|x^*_i - x_i| < \alpha \, Err/k$$

Part II

Paper	R/ D	Sketch length	Encode time	Column sparsity	Recovery time	Approx
[BGIKS'08]	D	k log(n/k)	n log(n/k)	log(n/k)	n^c	l1 / l1
[IR'08], [BIR'08],[BI'09]	D	k log(n/k)	n log(n/k)	log(n/k)	n log(n/k)* log	l1 / l1



vs.



- Restricted Isometry Property (RIP) [Candes-Tao'04]

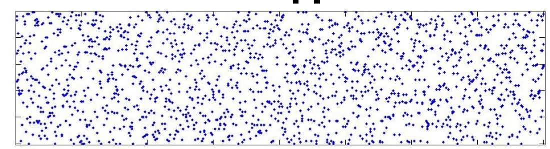
$$\Delta \text{ is } k\text{-sparse} \Rightarrow \|\Delta\|_2 \leq \|A\Delta\|_2 \leq C\, \|\Delta\|_2$$

- Holds w.h.p. for:
  - Random Gaussian/Bernoulli: $m=O(k \log (n/k))$
  - Random Fourier: $m=O(k \log^{O(1)} n)$
- Consider $m \times n$ 0-1 matrices with d ones per column
- Do they satisfy RIP ?
  - No, unless $m=\Omega(k^2)$ [Chandar'07]
- However, they can satisfy the following RIP-1 property [Berinde-Gilbert-Indyk-Karloff-Strauss'08]:

$$\Delta \text{ is } k\text{-sparse} \Rightarrow d\,(1-\varepsilon)\, \|\Delta\|_1 \leq \|A\Delta\|_1 \leq d\|\Delta\|_1$$

- Sufficient (and necessary) condition: the underlying graph is a $(\, k,\ d(1-\varepsilon/2)\,)$-expander

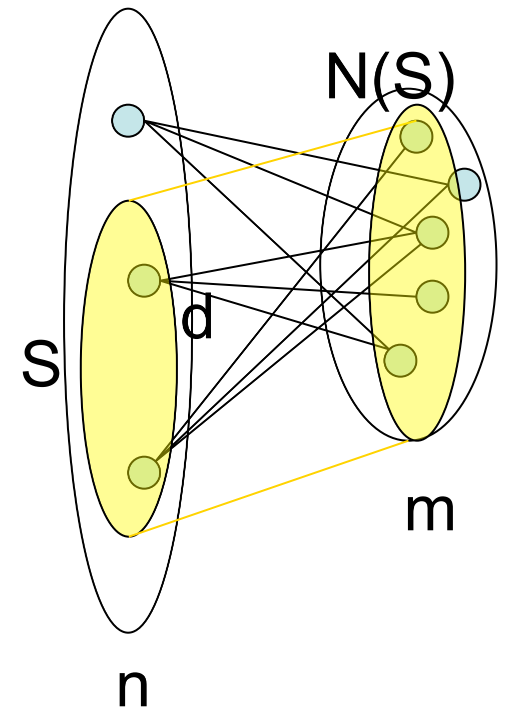
Expanders



- A bipartite graph is a $(k,d(1-\varepsilon))$-expander if for any left set S, $|S|\leq k$, we have $|N(S)|\geq(1-\varepsilon)d\,|S|$
- Objects well-studied in theoretical computer science and coding theory
- Constructions:
  - Probabilistic: $m=O(k \log (n/k))$
  - Explicit: $m=k$ quasipolylog n
- High expansion implies RIP-1:

  Δ is k-sparse $\Rightarrow d(1-\varepsilon)\,||\Delta||_1 \leq ||A\Delta||_1 \leq d||\Delta||_1$

  [Berinde-Gilbert-Indyk-Karloff-Strauss'08]



Proof: $d(1-\varepsilon/2)$-expansion $\Rightarrow$ RIP-1

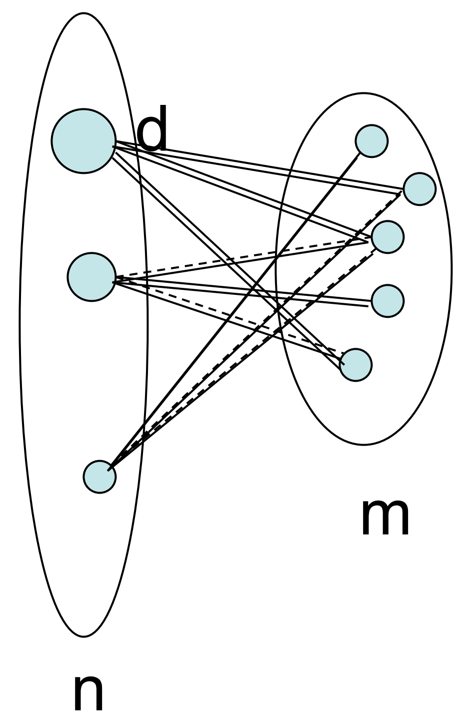
- Want to show that for any k-sparse Δ we have

$$d\,(1-\varepsilon)\,\|\Delta\|_1 \le \|A\,\Delta\|_1 \le d\|\Delta\|_1$$

- RHS inequality holds for ***any*** Δ
- LHS inequality:
  - W.l.o.g. assume

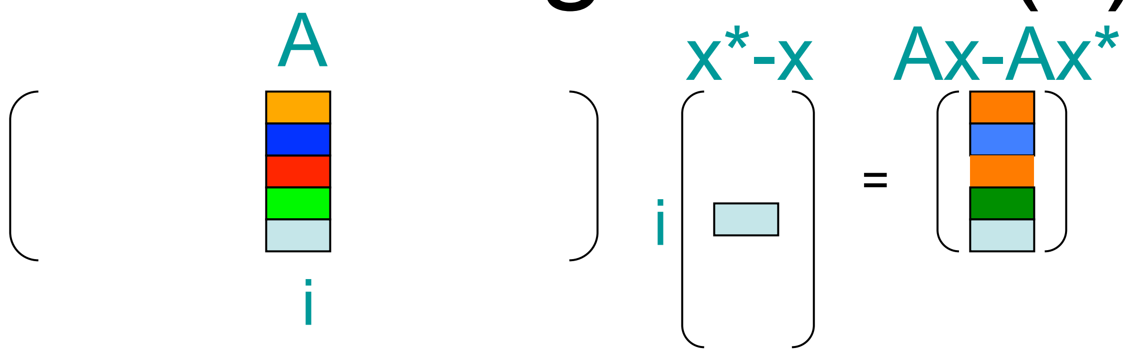
$$|\Delta_1| \ge \ldots \ge |\Delta_k| \ge |\Delta_{k+1}| = \ldots = |\Delta_n| = 0$$

  - Consider the edges $e=(i,j)$ in a lexicographic order
  - For each edge $e=(i,j)$ define $r(e)$ s.t.
    - $r(e)=-1$ if there exists an edge $(i',j)<(i,j)$
    - $r(e)=1$ if there is no such edge
- Claim 1: $\|A\Delta\|_1 \ge \sum_{e=(i,j)} |\Delta_i| r_e$
- Claim 2: $\sum_{e=(i,j)} |\Delta_i| r_e \ge (1-\varepsilon)\, d\|\Delta\|_1$



Recovery: algorithms

Matching Pursuit(s)

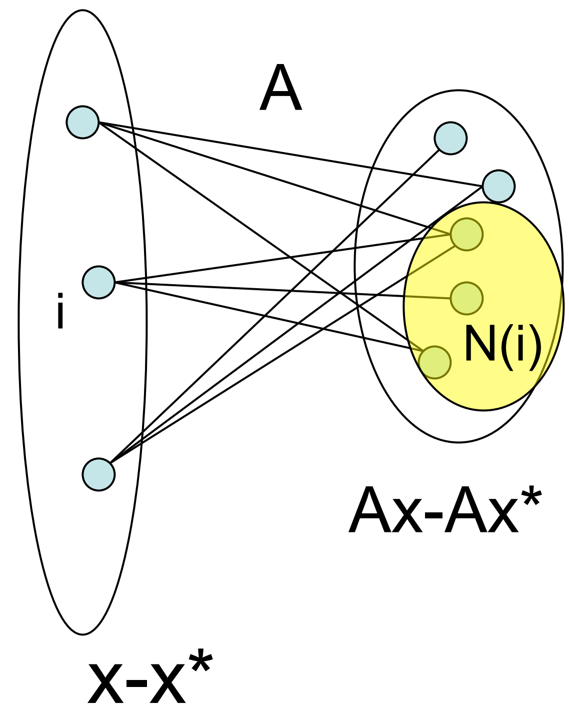


- Iterative algorithm: given current approximation x^* :
  - Find (possibly several) i s. t. A_i "correlates" with $Ax\text{-}Ax^*$. This yields i and z s. t.

    $$||x^*+ze_i-x||_p << ||x^* - x||_p$$
  - Update x^*
  - Sparsify x^* (keep only k largest entries)
  - Repeat
- Norms:
  - $p=2$: CoSaMP, SP, IHT etc (RIP)
  - $p=1$: SMP, SSMP (RIP-1)
  - $p=0$: LDPC bit flipping (sparse matrices)

Sequential Sparse Matching Pursuit

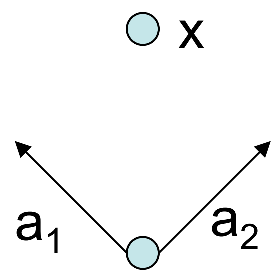
- Algorithm:
  - $x^*=0$
  - Repeat T times
    - Repeat $S=O(k)$ times
      - Find i and z that minimize* $\|A(x^*+ze_i)-Ax\|_1$
      - $x^* = x^*+ze_i$
    - Sparsify x^*
      (set all but k largest entries of x^* to 0)
- Similar to SMP, but updates done sequentially

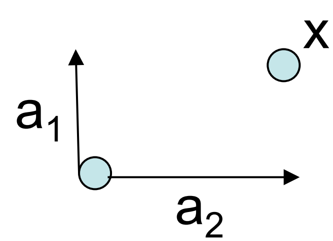


* Set $z=\text{median}[\,(Ax^*-Ax)_{N(I)}]$.Instead, one could first optimize (gradient) i and then z [Fuchs'09]

SSMP: Approximation guarantee

- Want to find k-sparse x^* that minimizes $||x-x^*||_1$
- By RIP1, this is approximately the same as minimizing $||Ax-Ax^*||_1$
- Need to show we can do it *greedily*





Supports of a_1 and a_2 have small overlap (typically)

Conclusions

- Sparse approximation using sparse matrices
- State of the art: deterministically can do 2 out of 3:
  - Near-linear encoding/decoding
  - O(k log (n/k)) measurements
  - Approximation guarantee with respect to L2/L1 norm

  This talk
- Open problems:
  - 3 out of 3 ?
  - Explicit constructions ?
    - Expanders (i.e., RIP-1 property)
    - Matrices with RIP property
      - Recent construction yields $O(k^{2-a})$ measurements for some a>0 and certain range of k [Bourgain, Dilworth, Ford, Konyagin, Kutzarova'10]

References

- Survey:

  A. Gilbert, P. Indyk, “Sparse recovery using sparse matrices”, Proceedings of IEEE, June 2010.
- Courses:
  - “Streaming, sketching, and sub-linear space algorithms”, Fall’07
  - “Sub-linear algorithms” (with Ronitt Rubinfeld), Fall’10
- Blogs:
  - Nuit blanche: nuit-blanche.blogspot.com/

Appendix

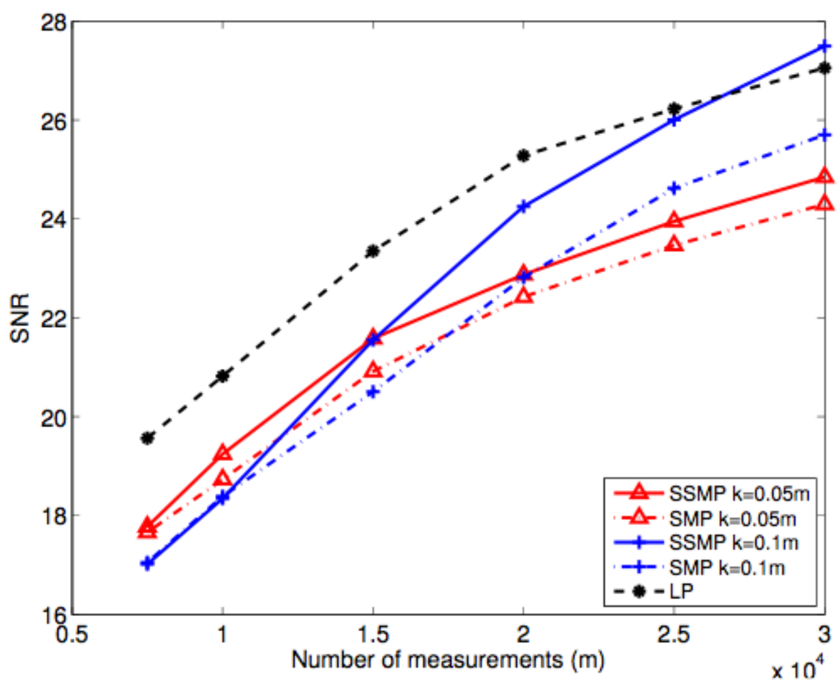
l_∞/l_1 implies l_1/l_1

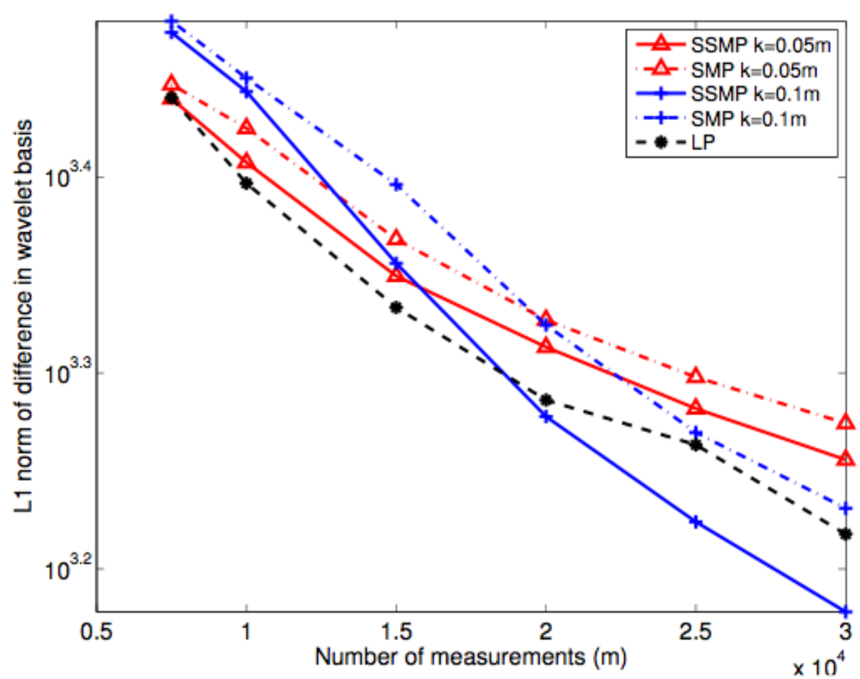
- Algorithm:
  - Assume we have x^* s.t. $||x-x^*||_\infty \leq C\ Err_1\ /k$.
  - Let vector x' consist of k largest (in magnitude) elements of x^*
- Analysis
  - Let S (or S^*) be the set of k largest in magnitude coordinates of x (or x^*)
  - Note that $||x^*_S|| \leq ||x^*_{S^*}||_1$
  - We have

$$
\begin{aligned}
||x-x'||_1 &\leq ||x||_1 - ||x_{S^*}||_1 + ||x_{S^*}-x^*_{S^*}||_1 \\
&\leq ||x||_1 - ||x^*_{S^*}||_1 + 2||x_{S^*}-x^*_{S^*}||_1 \\
&\leq ||x||_1 - ||x^*_S||_1 + 2||x_{S^*}-x^*_{S^*}||_1 \\
&\leq ||x||_1 - ||x_S||_1 + ||x^*_S-x_S||_1 + 2||x_{S^*}-x^*_{S^*}||_1 \\
&\leq Err + 3\alpha/k * k \\
&\leq (1+3\alpha)Err
\end{aligned}
$$

Experiments

256x256





SSMP is ran with S=10000,T=20. SMP is ran for 100 iterations. Matrix sparsity is d=8.

SSMP: Running time

- Algorithm:
  – $x^*=0$
  – Repeat T times
    - For each $i=1\ldots n$ compute* z_i that achieves

      $$D_i=\min_z \|A(x^*+ze_i)-b\|_1$$

      and store D_i in a heap
    - Repeat $S=O(k)$ times
      – Pick i,z that yield the best gain
      – Update $x^* = x^*+ze_i$
      – Recompute and store $D_{i'}$ for all i' such that $N(i)$ and $N(i')$ intersect
    - Sparsify x^*
      (set all but k largest entries of x^* to 0)
- Running time:

$$T\,[\,n(d+\log n) + k\, nd/m \cdot d\,(d+\log n)]$$

$$= T\,[\,n(d+\log n) + nd\,(d+\log n)] = T\,[\,nd\,(d+\log n)]$$