Orthogonal Range Queries

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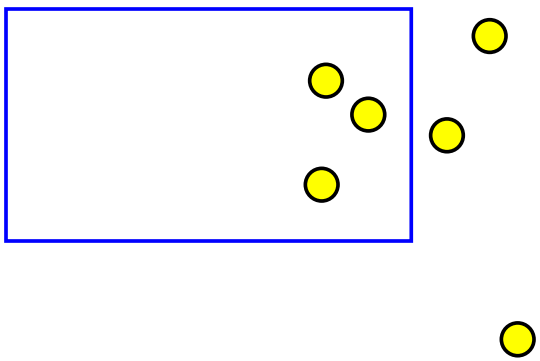
Range Searching in 2D

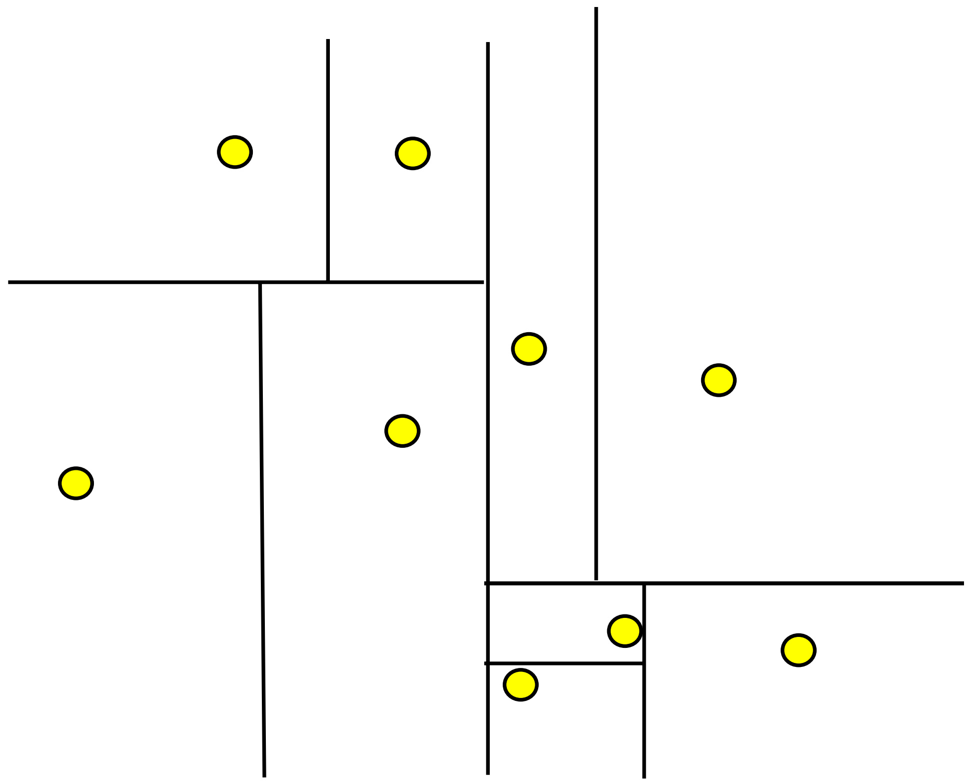
- Given a set of n points, build a data structure that for any query rectangle R, reports all points in R



Kd-trees [Bentley]

- Not the most efficient solution in theory
- Everyone uses it in practice
- Algorithm:
  - Choose x or y coordinate (alternate)
  - Choose the median of the coordinate; this defines a horizontal or vertical line
  - Recurse on both sides
- We get a binary tree:
  - Size: $O(N)$
  - Depth: $O(\log N)$
  - Construction time: $O(N \log N)$

Kd-tree: Example

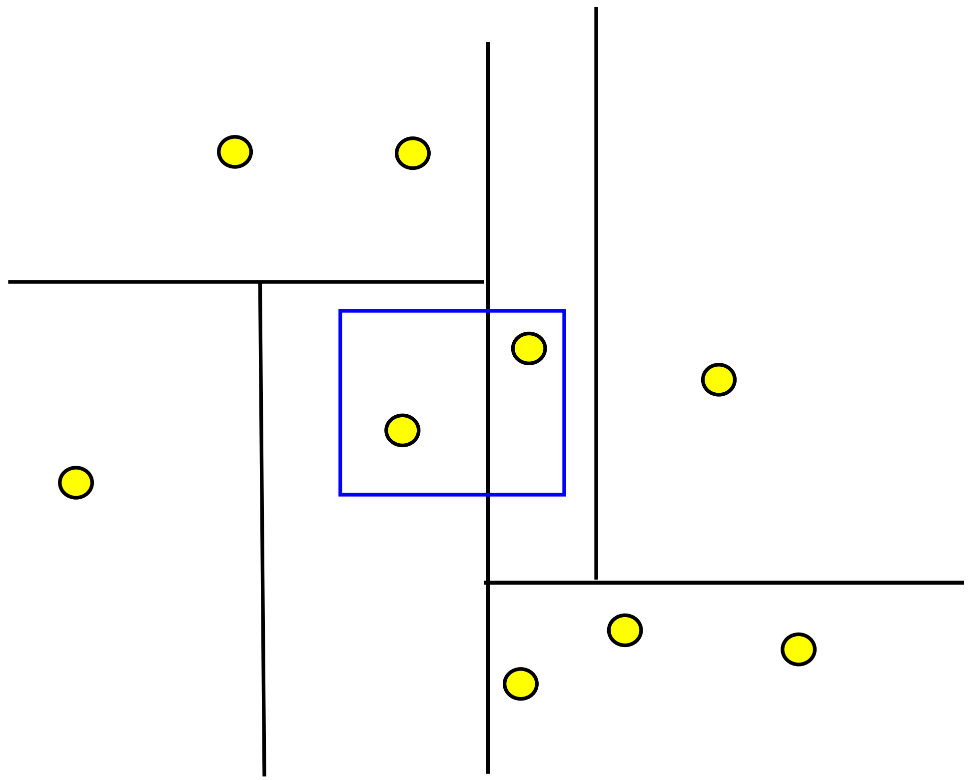


Each tree node v corresponds to a region Reg(v).

Kd-tree: Range Queries

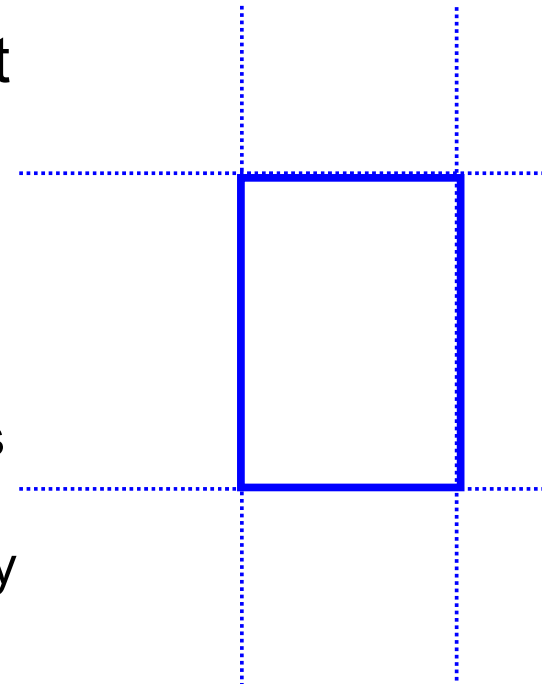
1. Recursive procedure, starting from v=root
2. Search (v,R):
   a) If v is a leaf, then report the point stored in v if it lies in R
   b) Otherwise, if Reg(v) is contained in R, report all points in the subtree of v
   c) Otherwise:
      - If Reg(left(v)) intersects R, then Search(left(v),R)
      - If Reg(right(v)) intersects R, then Search(right(v),R)

Query demo



Query Time Analysis

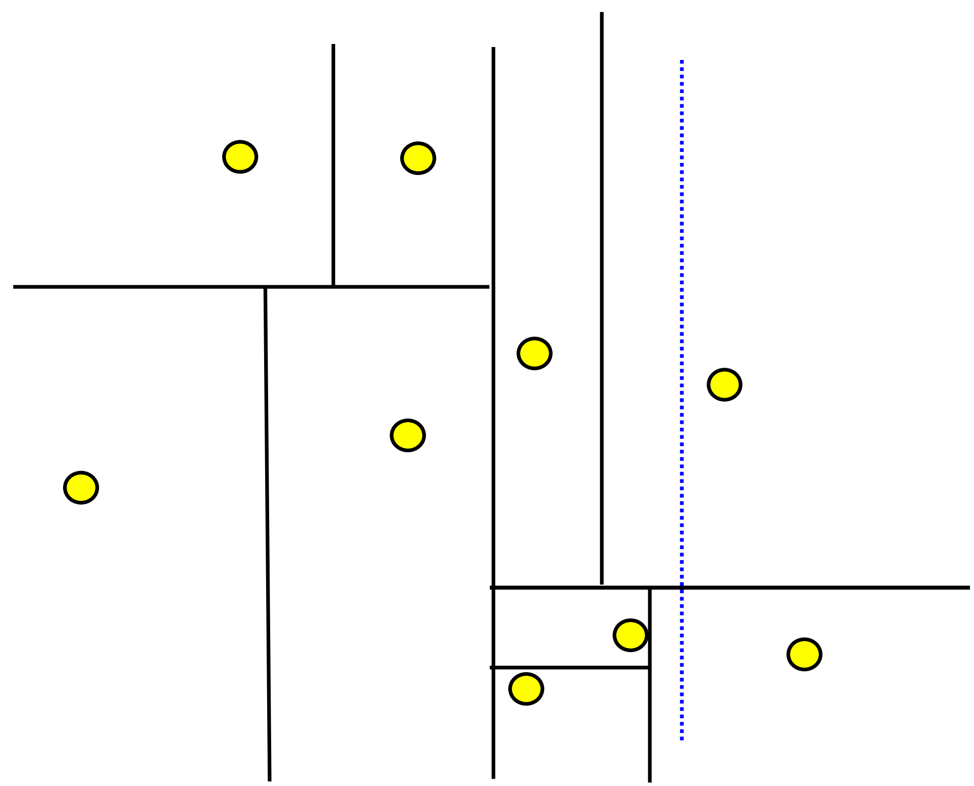
- We will show that Search takes at most $O(n^{1/2}+P)$ time, where P is the number of reported points
  - The total time needed to report all points in all sub-trees (i.e., taken by step b) is $O(P)$
  - We just need to bound the number of nodes v such that $Reg(v)$ intersects R but is not contained in R. In other words, the boundary of R intersects the boundary of $Reg(v)$
  - Will make a gross overestimation: will bound the number of $Reg(v)$ which are crossed by any of the 4 horizontal/vertical lines



Query Time Continued

- What is the max number $Q(n)$ of regions in an n-point kd-tree intersecting (say, vertical) line ?
  - If we split on x, $Q(n)=1+Q(n/2)$
  - If we split on y, $Q(n)=1+2*Q(n/2)$
  - Since we alternate, we can write $Q(n)=2+2Q(n/4)$
- This solves to $O(n^{1/2})$

Analysis demo

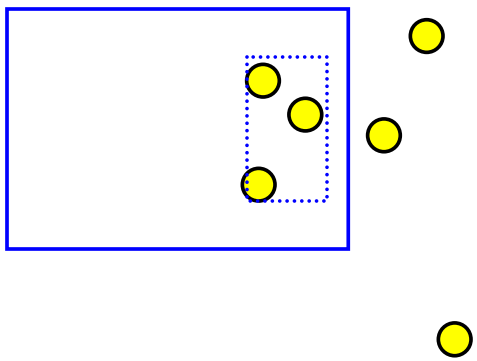


A Faster Solution

- Query time: $O(\log^2 n + P)$
- Space: $O(n \log n)$

Idea I: Ranks

- Sort x and y coordinates of input points
- For a rectangle $R=[x_1,x_2]\times[y_1,y_2]$, we have point $(u,v)\in R$ iff
  - $succ_x(x_1) \le rank_x(u) \le pred_x(x_2)$
  - $succ_y(y_1) \le rank_y(v) \le pred_y(y_2)$
- Thus we can replace
  - Point coordinates by their rank
  - Query boundaries by succ/pred; this adds $O(\log n)$ to the query time

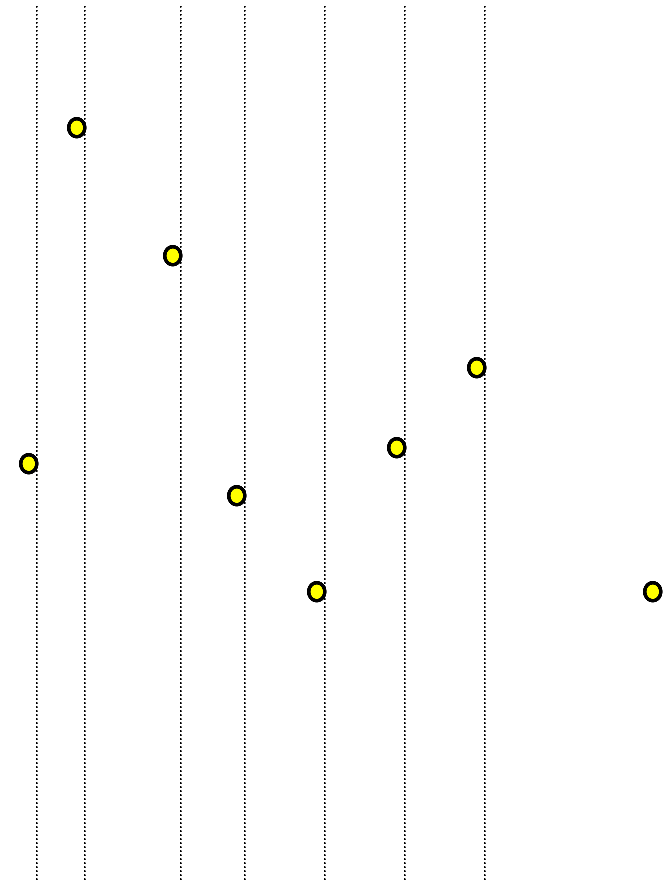


Dyadic intervals

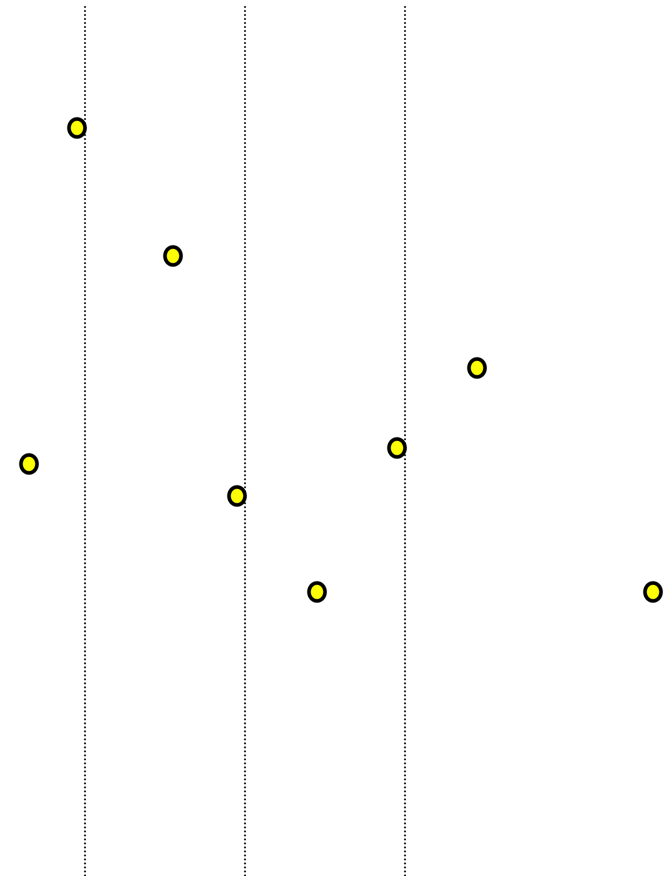
- Assume n is a power of 2. Dyadic intervals are:
  - [1,1] , [2,2] … [n,n]
  - [1,2] , [3,4] … [n-1,n]
  - [1,4] , [5,8] … [n-3,n]
  - ….
  - [1…n]
- Any interval {a…b} can be decomposed into O(log n) dyadic intervals:
  - Imagine a full binary tree over {1…n}
  - Each node corresponds to a dyadic interval
  - Any interval {a…b} can be "covered" using O(log n) sub-trees

Range Trees

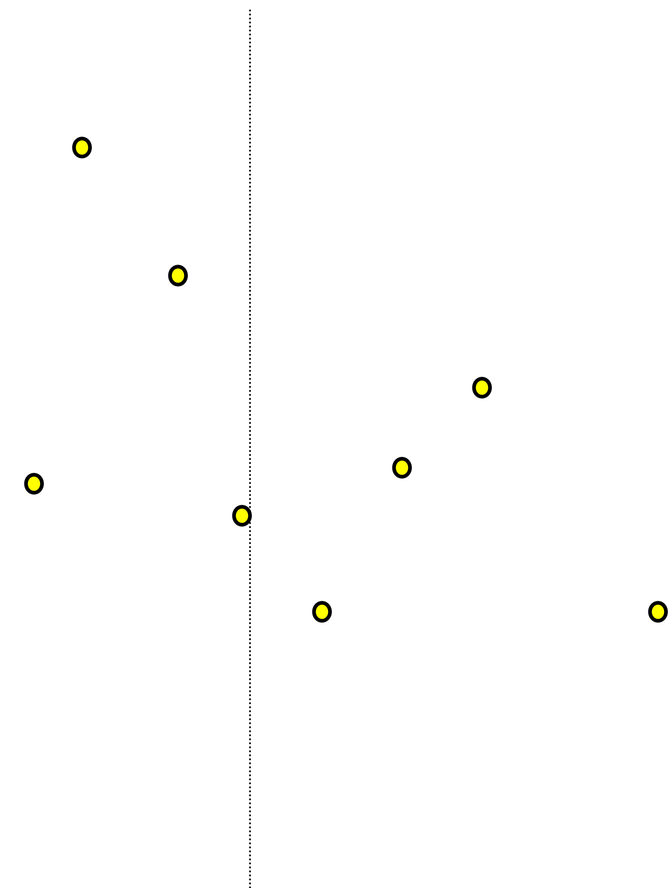
- For each level $l=1 \ldots \log n$, partition x-ranks using level-l dyadic intervals
- This induces vertical strips
- Within each strip, construct a BST on y-coordinates



Range Trees



Range Trees

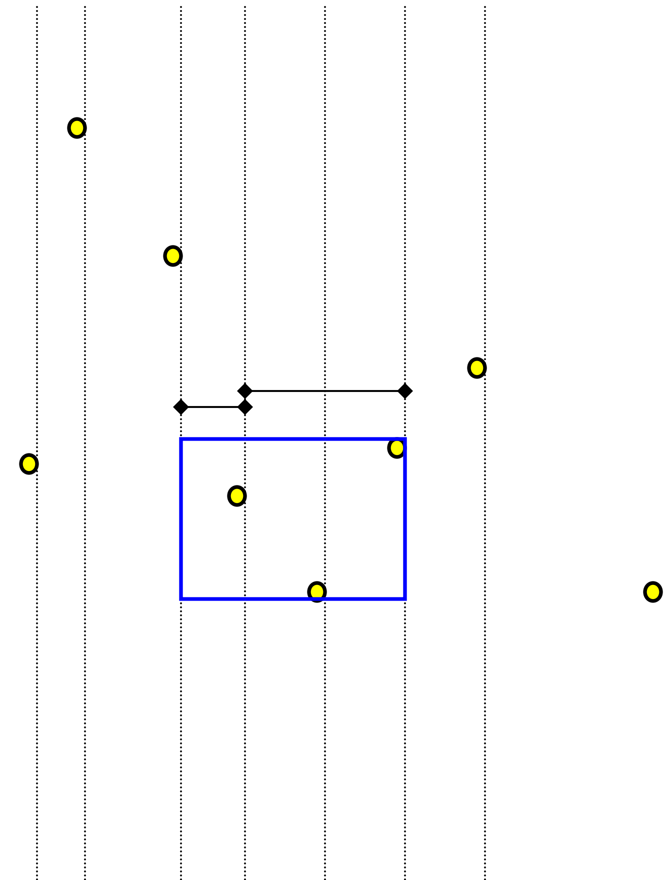


Analysis

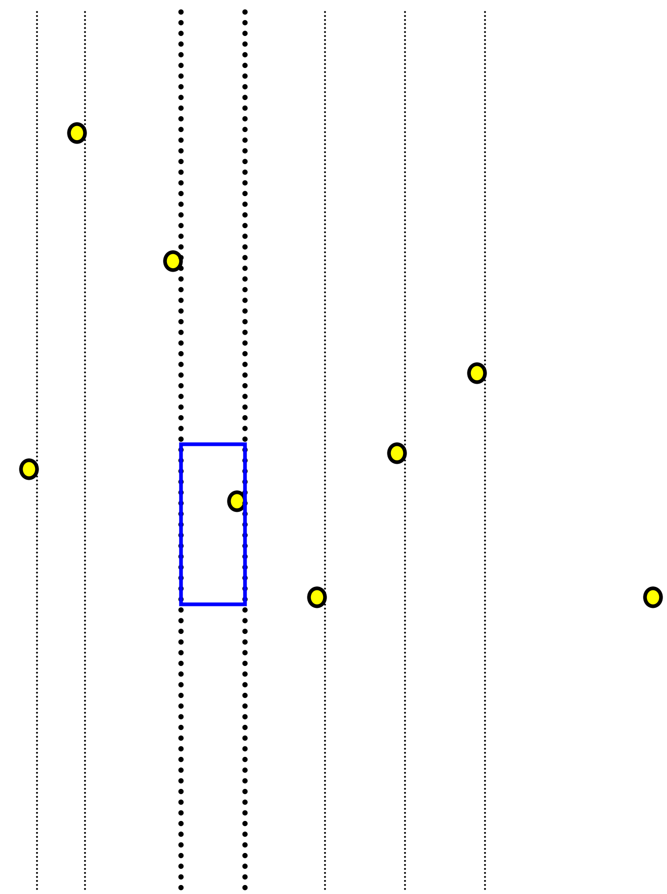
- Each point occurs in log n different levels
- Space: O(n log n)
- How do we implement the query ?

Query procedure

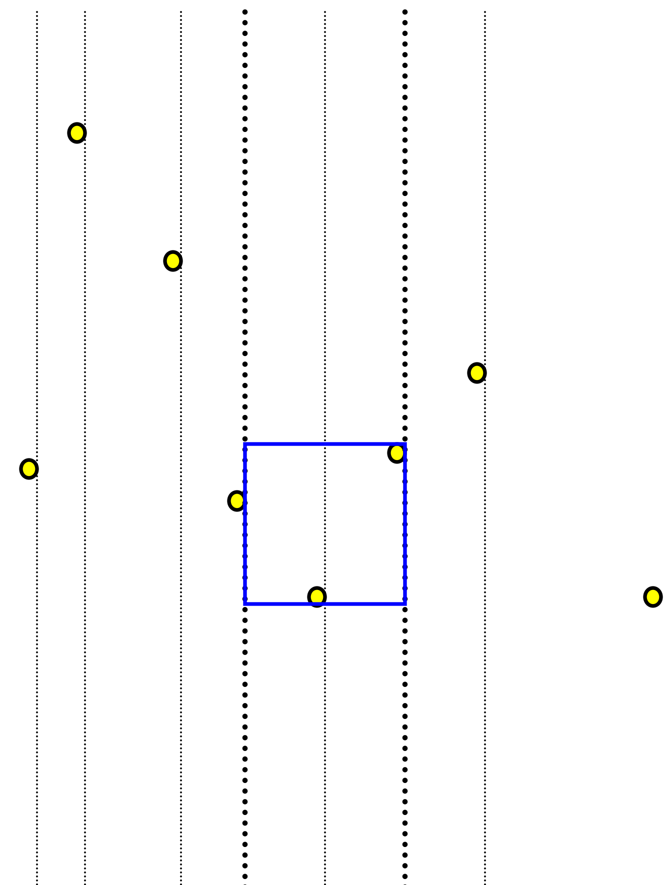
- Consider query $R = X \times Y$
- Partition X into dyadic intervals
- For each interval, query the corresponding strip BST using Y



Query procedure



Query procedure



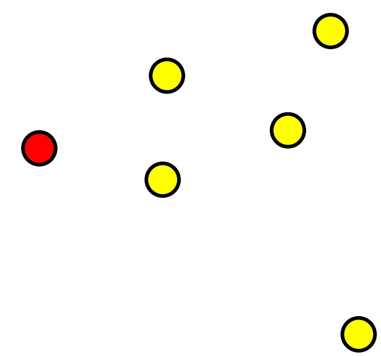
Analysis ctd.

- Query time:
  - $O(\log n + \text{output})$ time per strip
  - $O(\log n)$ strips
  - Total: $O(\log^2 n+P)$
- Faster than kd-tree, but space $O(n \log n)$
- Recursive application of the idea gives
  - $O(\log^d n)$ query time
  - $O(n \log^{d-1} n)$ space

  for the d-dimensional problem

Approximate Nearest Neighbor (ANN)

- Given: a set of points P in the plane
- Goal: given a query point q, and $\varepsilon > 0$, find a point p' whose distance to q is at most $(1+\varepsilon)$ times the distance from q to its nearest neighbor

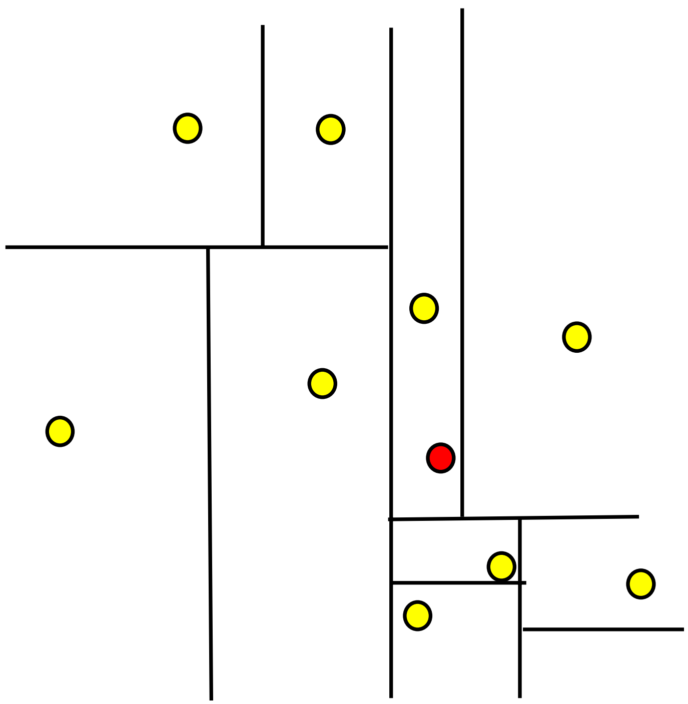


Our "solution"

- We will "solve" the problem using kd-trees…
- …under the assumption that all leaf cells of the kd-tree for P have bounded aspect ratio
- Assumption somewhat strict, but satisfied in practice for most of the leaf cells
- We will show
  - $O(\log n/\varepsilon^2)$ query time
  - $O(n)$ space (inherited from kd-tree)

ANN Query Procedure

- Locate the leaf cell containing q
- Enumerate all leaf cells C in the increasing order of distance from q (denote it by r)
  - Update p' so that it is the closest point seen so far
  - Note: r increases, dist(q,p') decreases
- Stop if $dist(q,p')<(1+\varepsilon)*r$



Analysis

- Running time:
  - All cells C seen so far (except maybe for the last one) have diameter > $\varepsilon * r$
  - …Because if not, then $p(C)$ would have been a $(1+\varepsilon)$-approximate nearest neighbor, and we would have stopped
  - The number of cells with diameter $\varepsilon * r$, bounded aspect ratio, and touching a ball of radius r is at most $O(1/\varepsilon^2)$