Geometric Optimization

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Geometric Optimization

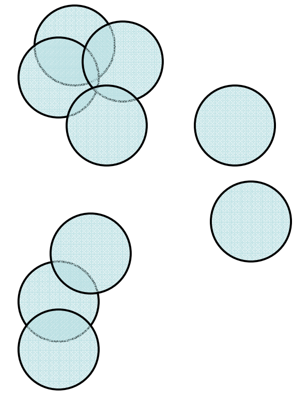
- Minimize/maximize something subject to some constraints
- Have seen:
  - Linear Programming
  - Minimum Enclosing Ball
  - Diameter/NN (?)
- All had easy polynomial time algorithms for d=2

Today

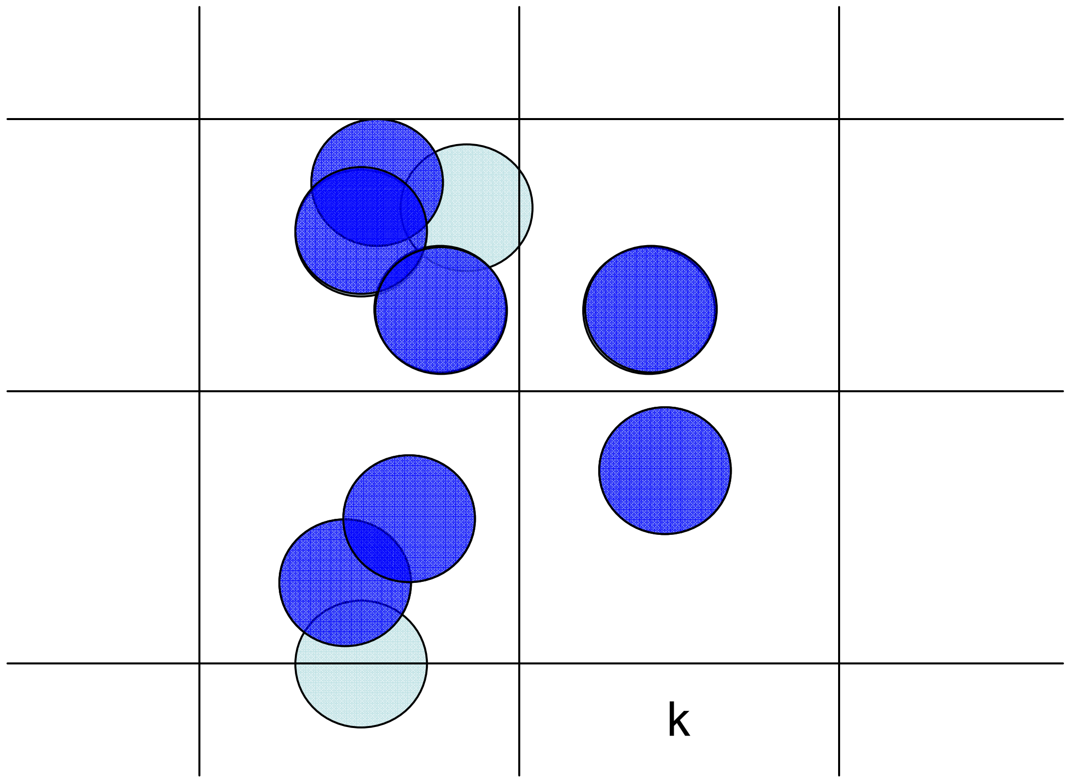
- NP-hard problems in the plane
  - Packing and piercing [Hochbaum-Maass'85]
  - TSP, Steiner trees, and a whole lot more [Arora'96] (cf. [Mitchell'96])
- Best exact algorithms exponential in n
- We will see $(1+\varepsilon)$-approximation algorithms that run in poly time for any fixed $\varepsilon>0$ (so, e.g., $n^{O(1/\varepsilon)}$ is OK)
- Such an algorithm is called a PTAS

Packing

- Given: a set C of unit disks $c_1 \ldots c_n$
- Goal: find a subset C' of C such that:
  - All disks in C' are pair-wise disjoint
  - $|C'|$ is maximized
- How to get a solution of size $\geq (1-\varepsilon)$ times optimum ?



Algorithm



Algorithm, ctd.

- Impose a grid of granularity k
- For each grid cell a
  - Compute C(a) = the set of disks fully contained in a
  - Solve the problem for C(a) obtaining C'(a)
- Report C' =union of C'(a) for all cells a

Computing c’(a)

- The grid cell a has side length = k
- Can pack at most k^2 disks into a
- Solve via exhaustive enumeration → running time $O(n^{k^2})$
- This also bounds the total running time

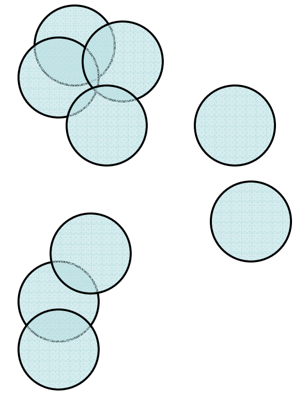
Analysis

- Approximation factor:
  - Consider the optimal collection of disks C
  - Shift the grid at random
  - The probability that a given $c \in C$ not fully contained in some a is at most

$$O(1)/k$$

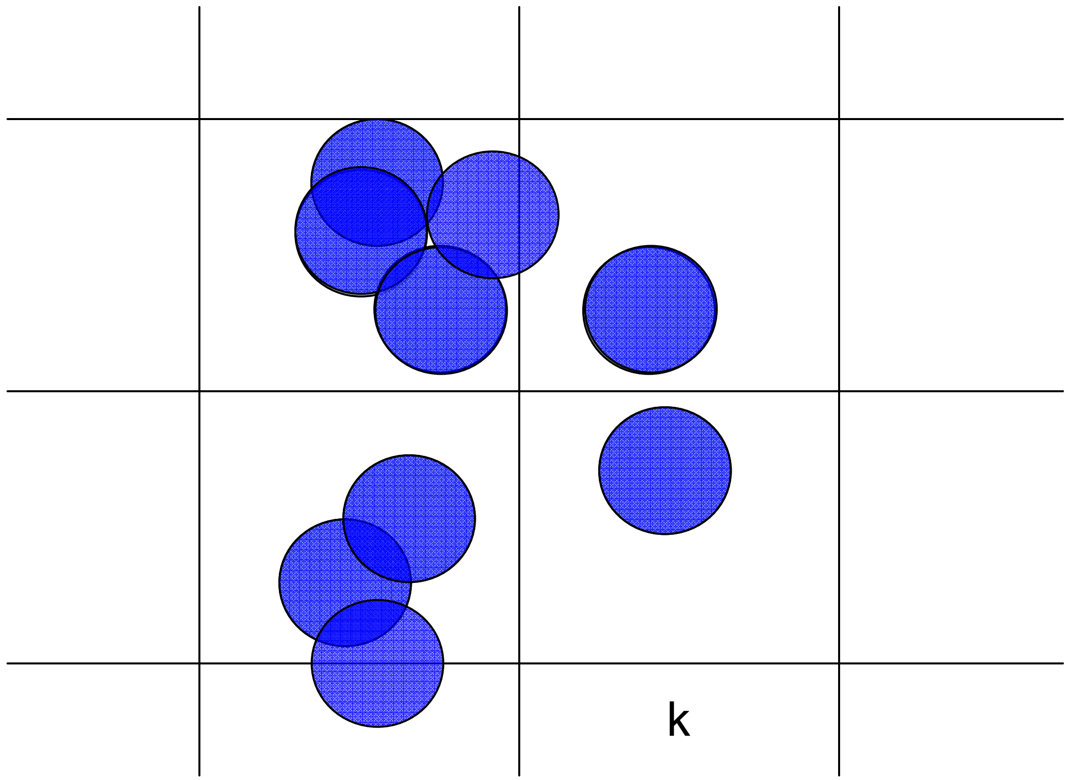
  - The expected number of removed disks is $O(|C|/k)$
  - Setting $k=O(1/\varepsilon)$ suffices
- Altogether: running time $n^{O(1/\varepsilon\text{^}2)}$

Piercing



- Given: a set C of unit disks $c_1 \ldots c_n$
- Goal: find a set P of points such that:
  - $P \cap c_i \neq \emptyset$ for $i=1\ldots n$
  - $|P|$ is minimized
- How to get a solution of size $\leq(1+\varepsilon)$ times optimum ?

Algorithm



Algorithm, ctd.

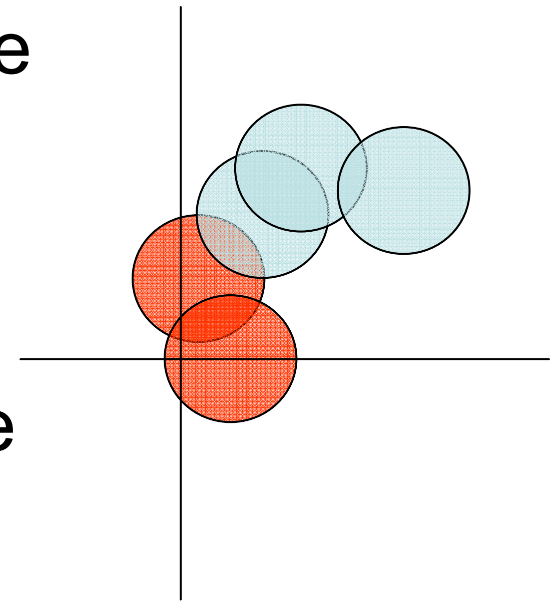
- Impose a grid G of granularity k
- For each cell a
  - Compute C(a) = the set of disks in C intersecting a
  - Solve the problem for C(a) obtaining P(a)
- Report P=union of P(a) for all cells a

Computing P(a)

- The grid cell a has side length = k
- Can pierce $C(a)$ using at most k^2 points
- Sufficient to consider $O(n^2)$ choices of piercing points (for all points in P)
  - Compute arrangement of disks in $C(a)$
  - Points in the same cell equivalent
- Solve via exhaustive enumeration → running time $O(n^{2k^2})$
- This also bounds the total running time

Analysis

- Problem: disks B within distance 1 from the grid boundary
- If we eliminated all the occurrences of the disks in B from all C(a)'s, one could pierce the remainder with cost at most OPT(C)
- Suffices to show that the cost of piercing B is small



More formally

- Define $B(a)=C(a) \cap B$
- The cost of our algorithm is at most

$$\sum_a OPT(C(a)) \le$$
$$\sum_a OPT(C(a)-B) + \sum_a OPT(B(a)) =$$
$$OPT(C-B) + \sum_a OPT(B(a))$$

- Also, $\sum_a OPT(B(a)) \le 4\ OPT(B)$, since
  - Take the optimal piercing T of B
  - Define $T(a)= T \cap B(a)$
  - We have $OPT(B(a)) \le |T(a)|$
  - Also, each element of T appears in ≤ 4 sets $T(a)$
  - Thus $\sum_a OPT(B(a)) \le \sum_a |T(a)| \le 4\ |T| = 4\ OPT(B)$

Analysis ctd.

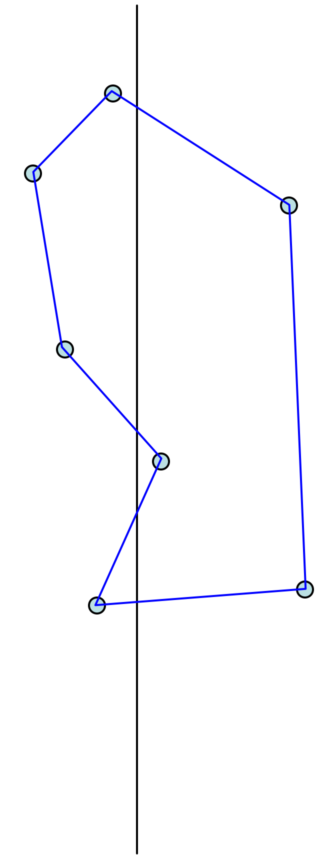
- Suffices to show that the cost OPT(B) of optimally piercing B is $O(\varepsilon\ OPT(C))$
- Let P be the optimum piercing of C
- Shift the grid at random
- All disks in B are pierced by

$$P' = P \cap (G \oplus Ball(0,2))$$

- The probability that a fixed $p \in P$ belongs to P' is $O(1)/k$
- The expected |P'| of is $O(|P|)/k$
  - Setting $k=O(1/\varepsilon)$ suffices
- Altogether: running time $n^{O(1/\varepsilon^2)}$

Dynamic programming for TSP

- Divide the points using randomly shifted line
- Enumerate "all" possible configurations C of crossing points
- For each C, solve the two recursive problems
- Choose the C that minimizes the cost



Analysis

- Structure lemma: for any tour T, there is another tour T' with cost not much larger than the cost of T, which has only constant number of crossing points.