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Rigid Flattening of Polyhedra with Slits

Zachary Abel, Robert Connelly, Erik D. Demaine, Martin L. Demaine,
Thomas C. Hull, Anna Lubiw, and Tomohiro Tachi

Abstract. Cauchy showed that if the faces of a convex polyhedron are rigid then the whole polyhedron is rigid. Connelly showed that this is true even if finitely many extra creases are added. However, cutting the surface of the polyhedron destroys rigidity and may even allow the polyhedron to be flattened. We initiate the study of how much the surface of a convex polyhedron must be cut to allow continuous flattening with rigid faces. We show that a regular tetrahedron with side lengths 1 can be continuously flattened with rigid faces after cutting a slit of length .046 and adding a few extra creases.

1. Introduction

In many real-life situations we want polyhedra or polyhedral surfaces to flatten—think of paper bags, cardboard boxes, and foldable furniture. Although paper is flexible and can bend and curve, materials such as cardboard, metal, and plastic are not. The appropriate model for such non-flexible surfaces is "rigid origami" where the polyhedral faces are rigid and folding occurs only along pre-defined creases. In rigid origami, flattening is not always possible, and in fact, often no movement is possible at all. In particular, Cauchy's theorem of 1813 says that if a convex polyhedron is made with rigid faces hinged at the edges then no movement is possible (see [6]). Connelly [4] showed that this is true even if finitely many extra creases are added.

However, cutting the surface of the polyhedron destroys rigidity and may even allow the polyhedron to be flattened. For example, a paper bag is a box whose top face has been removed, so the afore-mentioned rigidity results do not apply. Everyone knows the "standard" folds for flattening a paper bag. Surprisingly, these folds do not allow flattening with rigid faces unless the bag is short [2]. Taller bags can indeed be flattened with rigid faces, but a different crease pattern is required [9]. Many of the clever ways of flattening cardboard boxes involve not only removal of the top face, but also extra slits and interlocking flaps in the bottom face.

In this paper we initiate the study of "rigid flattening" of a polyhedron: continuous flattening with rigid faces after the addition of finitely many cuts and creases.

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We require that the final flat folding be a flat folding of the original polyhedron, i.e. that every cut closes up at the end of the flattening process.

We can use previous results to show that every convex polyhedron has a rigid flattening. Without the requirement about cuts closing up, we could just cut every edge of the polyhedron and move the faces to the plane. Alternatively, we could keep the surface connected and use the "continuous blooming" of the source unfolding of a convex polyhedron [5].

With our requirement that cuts close up, the final state is a flat folding of the original polyhedron, so we first need to know that every convex polyhedron has a flat folded state. There are three proofs of this result: Bern and Hayes [3], using a disk-packing method that applies to any polyhedral surface; Itoh et al. [8] via a continuous motion; Abel et al. [1] via an easily-computable continuous motion resulting in a flat folding that respects the straight-skeleton gluing. Using these results, we can obtain a rigid flattening by just cutting every fold in the flat folded state. Note that the surface becomes disconnected.

It is an open question whether every convex polyhedron has a rigid flattening using cuts that do not disconnect the surface. More generally, we might ask to minimize the length of the cuts. Another interesting question is whether there is a rigid flattening with only one degree of freedom.

In this paper we begin exploring these ideas by studying the regular tetrahedron. We show that a surprisingly small cut allows rigid flattening. Specifically, if the tetrahedron has side length 1, a cut of length .046 suffices. We explicitly specify the few extra creases that are needed. There is one degree of freedom during the flattening. We use Mathematica to model the motion and verify that no self-intersections occur.

We argue that our particular slit cannot be reduced in length, but it is possible that a smaller slit in a different position works. In fact it is even possible that the slit length can approach 0 while the number of creases grows. We discuss these and other open questions in the final section of the paper.

2. Flattening a Regular Tetrahedron

In this section we show that a regular tetrahedron with side length 1 can be rigidly flattened with a cut of length .046. We specify the cut and the extra creases, and verify in Mathematica that the result folds flat rigidly without self-intersections.

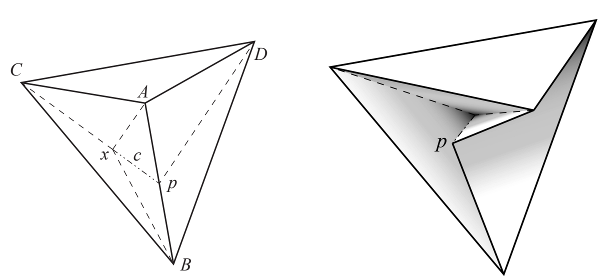

Figure 1. Flat folding of a regular tetrahedron: (*left*) the creases; (*right*) part way towards the flat folding (shown schematically, since the faces will not really be flat in this configuration).

In order to describe the cut and the extra creases we first explore a rigid flattening using a longer cut. The most natural flattening of a regular tetrahedron on vertices A, B, C, D uses creases as shown in Figure 1: faces ACD and BDC are intact; face ADB has one crease bisecting the angle at D and arriving at point p of the opposite edge AB; and the final face ABC has four creases to its centroid x—three from the vertices and the fourth crease, c, from p to x. We call c the *centroid normal*. All the creases are valley folds except c which is a mountain fold.

This flat folding yields a rigid flattening if we cut the centroid normal c and add two mountain creases that go from the vertices A and B to the cut c and bisect the angles $\angle xAp$ and $\angle xBp$ respectively. See Figure 2. (In fact, mountain creases from A and B to the midpoint of c would also work.) This rigid flattening was first shown by Connelly [4].

We argue that this flattening has only one Degree of Freedom (DOF). Suppose face BCD is fixed in 3-space. Faces ACD and BCD are rigid and the one degree of freedom is the angle between them. Given a value for that angle, the positions of x and p are fixed in 3-space. (The fact that Cx and Dp are valleys rules out the other possible position for each). This in turn fixes the positions of the final two mountain creases.

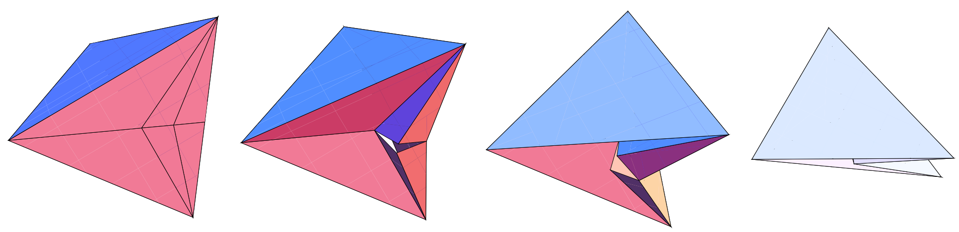

FIGURE 2. Rigid flattening of a tetrahedron after cutting the centroid normal c.

Our rigid flattening with a small slit is based on the one shown in Figure 2 but uses a shorter cut along the centroid normal. The cut goes from p, the midpoint of edge AB to a point q on the centroid normal. See Figure 3. The final length of pq will be .046, although we will discuss other possibilities. The triangles Apq and Bpq are called the *flaps*. Creases Ax, Bx, and Cx remain valleys. Crease xq is a mountain. Creases Aq and Bq will alternate between mountain and valley folds during the rigid flattening. Point m is placed on the centroid normal segment px and on the angle bisector of $\angle qAp$. We would like to add mountain creases Am and Bm, but this plan needs some refinement.

There are two limitations on the length of the cut pq. The first one can be remedied, but the second one is more fundamental and makes it impossible to shorten the cut below .046. The second limitation is described below in Subsection 2.1. Here we address the first limitation.

The first limitation is that as the slit becomes smaller the flaps can interfere with each other during the folding process. In particular, the two copies of m will collide in the rigid unfolding. This can be remedied by adding pleat folds to the flaps so that the two sides of the cut fold out of the way. In order to achieve the cut length of .046 we place pleat folds as shown in Figure 4. The pleat creases emanate from points p and q, with the the largest pleat crease at p forming angle $\angle qpr = 45°$

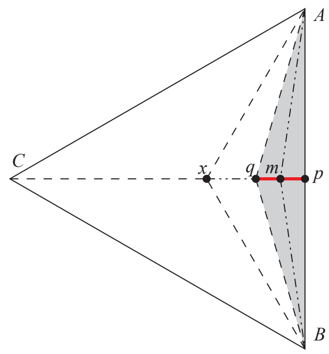

FIGURE 3. The plan of the creases to allow rigid flattening of a tetrahedron after cutting segment pq. (Other faces remain the same.) The flaps Apq and Bpq are shaded.

and the smaller pleat angles at p, going counter-clockwise in order, are 15°, 15°, 5°, 10° and then repeat in reverse order. This choice of angles is made so as to avoid collisions between the pleats across the slit when folding; merely quadrisecting the 45° angles at p causes collisions near the flat-folded state. Such pleats are effective in addressing the first limitation because they break the line Am (in Figure 3) so that instead of being a long mountain crease, it is now a shorter mountain, and then short valley-mountain-valley-mountain creases as we approach the slit. This makes Am contract into a zig-zag near the slit, which keeps the two copies of m away from each other during the folding process.

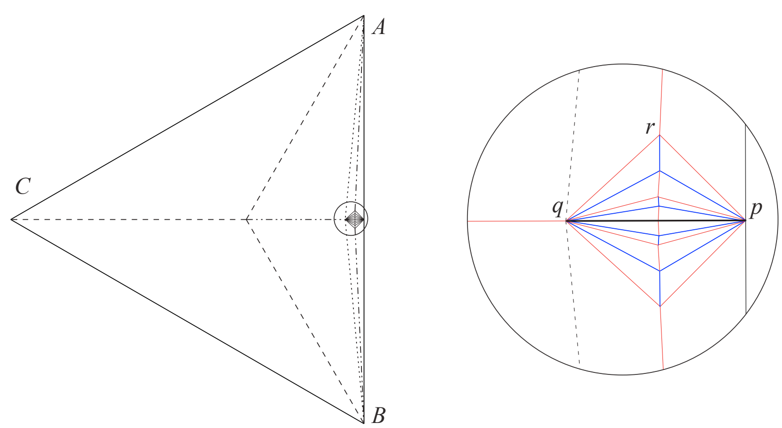

FIGURE 4. Crease pattern for rigid flattening of a regular tetrahedron after cutting a slit of length .046. (*left*) Crease pattern on face ABC. (*right*) close-up of the circled slit region (red lines are mountain folds, blue lines are valleys, and the dashed creases switch between the two).

Note that the unfolded creases from the point r to the center of the slit in Figure 4 (which are the same as the creases from x_3 to x_7 in Figure 5(a) below) form a slight zig-zag instead of a straight line. This is needed to ensure that the vertices of the pleats (x_3-x_7 in Figure 5(a)) will be flat-foldable. That is, the angles around these vertices must satisfy Kawasaki's Theorem, which states that the sum of the opposite pairs of angles at each vertex must sum to 180° in order to fold flat (see [7]).

This completes the description of the cut and the extra folds to enable rigid flattening of a regular tetrahedron with a slit length of .046. There is still one degree of freedom because the folding of the degree four vertices of the pleats will be determined by the neighboring creases adjacent to vertices A and B. Figure 6 shows 4 frames of the rigid flattening. Note in particular that frame 3 shows how the flaps have folded out of the way and avoided colliding. With careful observation, one can see that the fold Aq is a mountain in frame 3 and a valley in frame 4.

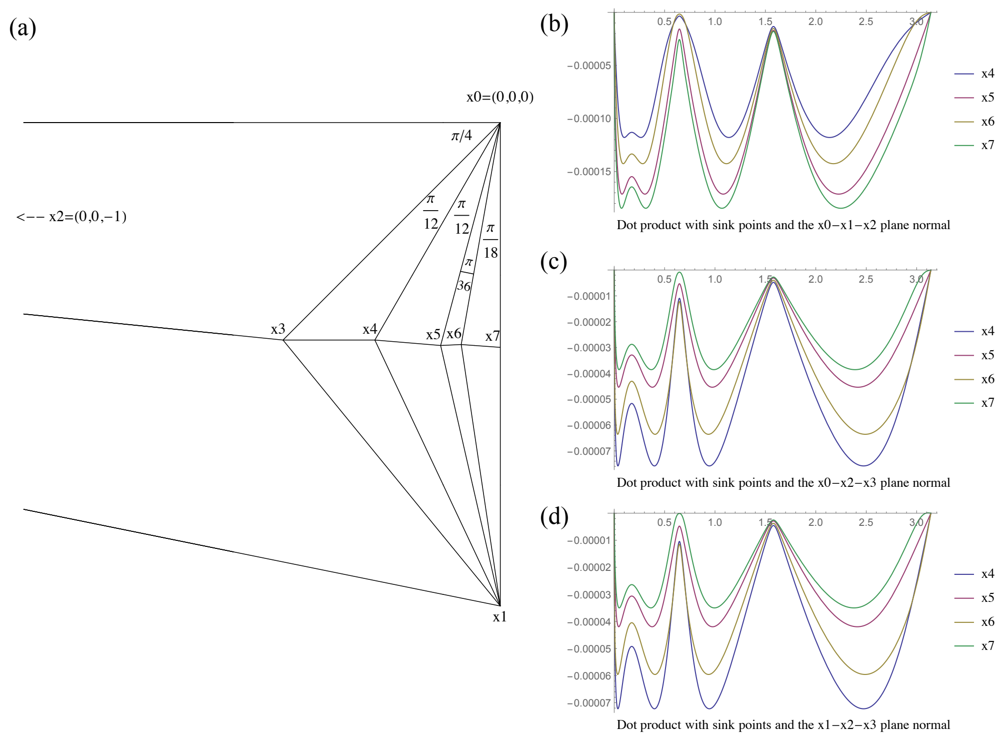

FIGURE 5. (a) Schematic of the pleat "sink" points. (b)-(d) Graphs of the dot product between the sink point vectors and the normals to the sides of the cone from x_2 to the triangle made by x_0, x_1, and x_3.

2.1. A limitation on the cut length. In this section we show that the cut length cannot be shorter than 0.046 if we place the cut and the creases as shown in Figure 3 and allow extra folds only in the flaps. Note that this is a very limited result. It is quite possible that there is a rigid flattening using a shorter cut in a different position, or even in the same position but with extra folds outside the flaps.

Consider the creases in Figure 3. We will ignore the flaps—just cut them out of the surface. The remaining surface consists of 8 rigid triangles: 4 on face ABC

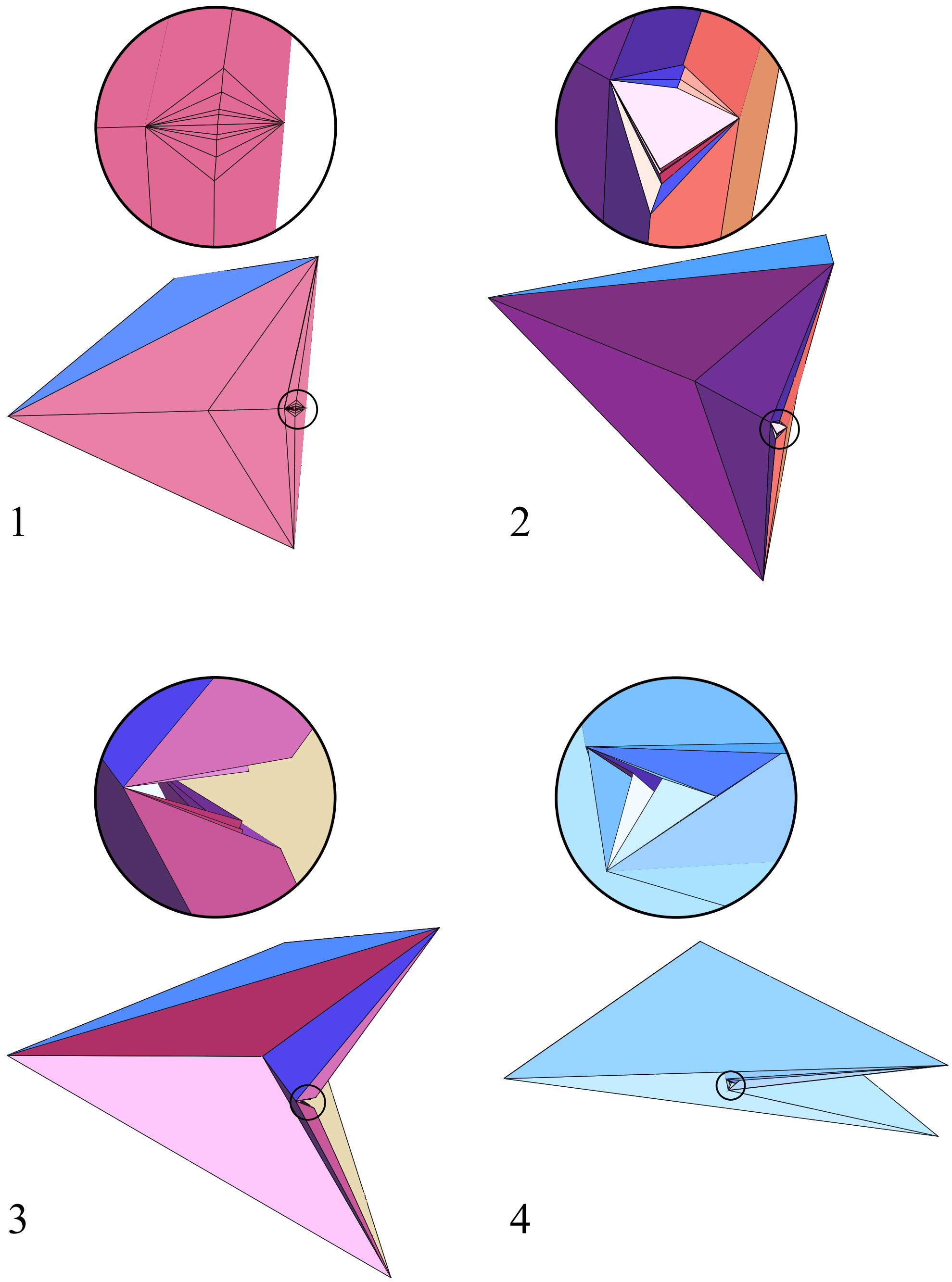

Figure 6. Rigid flattening of a regular tetrahedron after cutting a slit of length .046, showing 4 frames of the flattening, each with the detail of the slit region shown in the circular close-up.

of the tetrahedron, plus 2 on face ABD, plus the 2 intact faces. There is only one degree of freedom during the rigid flattening. Points C, D, x, p, q remain on the same plane. A cross-section in that plane is shown in Figure 7.

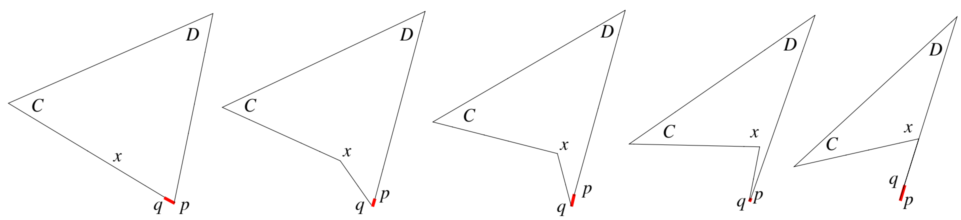

FIGURE 7. Cross sections in the plane of C, D, x showing how points p and q change distance during the rigid flattening process.

During the flattening process the distance between points p and q changes. Our main observation is that the cut pq must be long enough to accommodate this. In particular, if the flaps are included, they prevent p and q from being farther apart than the length of the cut. This is true even if the flaps are completely flexible.

We wrote a Mathematica program to compute the distance between points p and q in the plane of C, D, x during the rigid flattening of the 8 rigid triangles described above. As can be seen in Figure 8, if the cut length is less than 0.0461201 then at some point during the rigid flattening, points p and q will be further apart than the cut length. Thus the minimum possible cut is approximately 0.0461201.

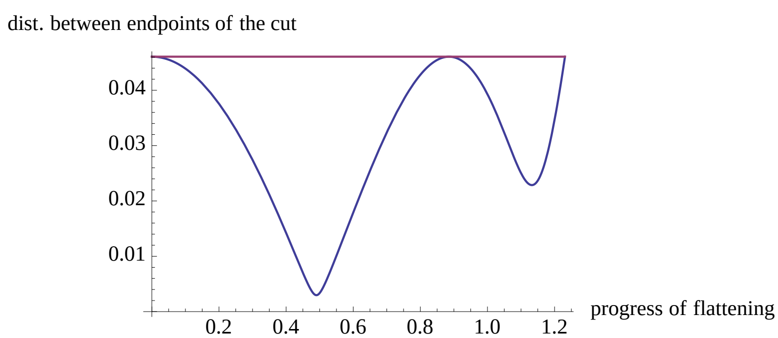

FIGURE 8. A graph of the distance between points p and q (y-axis) during the rigid flattening (x-axis) of the crease pattern in Figure 3 when the cut length is 0.0461201. Observe that the graph has a local maximum at an intermediate point of the flattening and this local maximum reaches the original cut length of 0.0461201 (indicated by the horizontal line). If we decrease the cut length, the new graph has a local maximum that exceeds the cut length.

2.2. Checking potential collisions. We have verified the rigid flattening of this model in Mathematica using a kinematics model for the regions between the creases. Figure 5(a) shows a detail of the pleats near the slit in the crease pattern, which we could also refer to as a *sink* to borrow origami terminology. The points x_0,

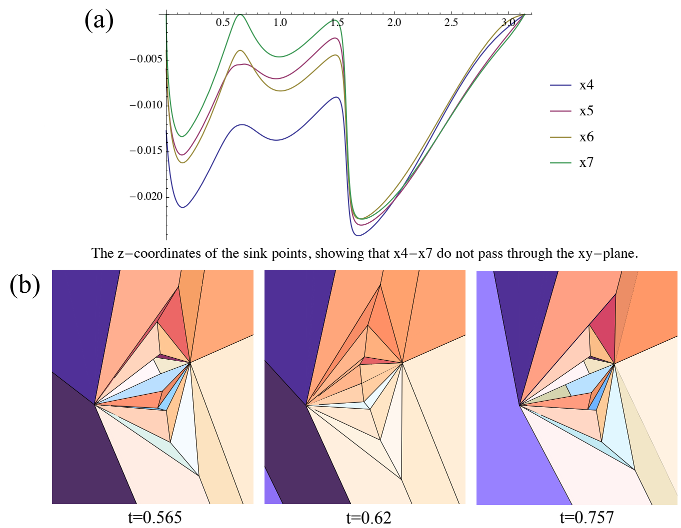

FIGURE 9. (a) Plots of the z-coordinate of the sink points in our Mathematica model for slit length 0.046. (b) Close-ups of the slit opening and closing near $t = 0.62$.

x_1, and x_2 correspond to the points p, q, and A from Figure 4, respectively, except that we imagine them to be in the xz-plane. To argue that the sink points x_4-x_7 do not collide with other parts of the model, we examine the polygonal cone made by x_2 (the cone point) and the triangle $x_0x_1x_3$. If the sink points remain inside this cone throughout the folding process then they will not collide with other sides of the folding tetrahedron. To this end, we graph the dot products of the outward-pointing vectors normal to the sides of this cone and the sink point vectors; so long as these dot products are nonpositive, the sink points will not penetrate the planes made by the cone sides and thus will remain inside the cone. Graphs of these dot products are shown in Figure 5(b), (c), and (d), where the horizontal axis is the folding angle $0 \leq t \leq \pi$, and the length of the slit is taken to be approx. 0.046. Note that the dot products all remain negative with the slight exception of the point x_7 in graph (d) at the end of the folding process ($t \approx \pi$). This is because when the cone is nearly flat the sink pleat made by x_7 is inclined upward and escapes the cone. However, examining this case shows that x_7 quickly folds flat and remains clear of any collisions as $t \to \pi$.

We also must show that none of the sink points collide with their mirror-image counterparts on the other side of the sink. As mentioned previously, if the angles of the sink pleats are not chosen with care, then such collisions will occur and obstruct the rigid folding of the model. One way to check this with our current model is to see if during the folding the sink points pass through the plane through $p = x_0$, C, and perpendicular to AB in Figure 4. In our Mathematica model, we had this plane be the xy-plane throughout the folding, so all we need to do is to plot the z-coordinates of the sink points x_4-x_7. This is shown in Figure 9(a), where the slit length is taken to be 0.046. Since the z-coordinates remain negative, the sink points will not collide with their counterparts on the other side of the slit. Note, however, that x_7 does touch its mirror-image at the folding angle $t \approx 0.62$, where the slit closes up before opening again, as seen in Figure 9(b).

Therefore, with the pleats in Figure 4 included, the two sides of the cut will not intersect throughout the folding process. Interested readers can download and examine the Mathematica code for this model at http://mars.wne.edu/~thull/rigidtet/tet.html.

3. Discussion and Open Problems

Our investigation of rigid flattening of a regular tetrahedron leaves many open questions:

(1) Does every convex polyhedron have a rigid flattening using cuts that leave the surface connected? Connelly [4] shows that cuts in the interiors of faces will not suffice.
(2) Is there such a rigid flattening with one degree of freedom?
(3) Does a regular tetrahedron with unit side lengths have a rigid flattening using a cut of length less than .046? Can the cut length approach 0 (as the number of extra creases grows)?
(4) Does every box have a rigid flattening using one straight cut? We suppose the answer is yes if you cut almost all the way around one equator and apply the rigid flattening for shallow paper bags [2], but will a shorter cut suffice?

Regarding the main open question (1), it may be easier to start from the flat folded state and ask what slits (plus extra creases) allow the flat folded state to unfold. Based on the example in Figure 2, where the regular tetrahedron is slit from the centroid of one face to an edge normal, we conjecture that it suffices to slice all the mountain folds in the flat folding. This is not true for polyhedral surfaces in general (the square twist is a counterexample), but might be true for convex polyhedra, or at least for straight skeleton flat foldings of convex polyhedra [1]. As a starting point, what happens if we cut the mountain folds *and* the edges of the polyhedron.

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Abel: Department of Mathematics, MIT, Cambridge MA, USA
E-mail address: `zabel@mit.edu`

Connelly: Department of Mathematics, Cornell University, Ithaca NY, USA
E-mail address: `rc46@cornell.edu`

Demaine, E.: MIT Computer Science and Artificial Intelligence Lab., Cambridge, USA
E-mail address: `edemaine@mit.edu`

Demaine, M.: MIT Computer Science and Artificial Intelligence Lab., Cambridge, USA
E-mail address: `mdemaine@mit.edu`

Hull: Department of Mathematics, Western New England U., Springfield MA, USA
E-mail address: `thull@wne.edu`

Lubiw: School of Computer Science, University of Waterloo, Waterloo ON, Canada
E-mail address: `alubiw@uwaterloo.ca`

Tachi: Department of General Systems Studies, The University of Tokyo, Japan
E-mail address: `tachi@idea.c.u-tokyo.acjp`