A generalization of the source unfolding of convex polyhedra

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Abstract. We present a new method for unfolding a convex polyhedron into one piece without overlap, based on shortest paths to a convex curve on the polyhedron. Our "sun unfoldings" encompass source unfolding from a point, source unfolding from an open geodesic curve, and a variant of a recent method of Itoh, O'Rourke, and Vîlcu.

Introduction

The easiest way to show that any convex polyhedron can be unfolded is via the *source unfolding* from a point s, where the polyhedron surface is cut at the *ridge tree* of points that have more than one shortest path to s; see [**2**]. The unfolding does not overlap because the shortest paths from s to every other point on the surface develop to straight lines radiating from s, forming a star-shaped unfolding.

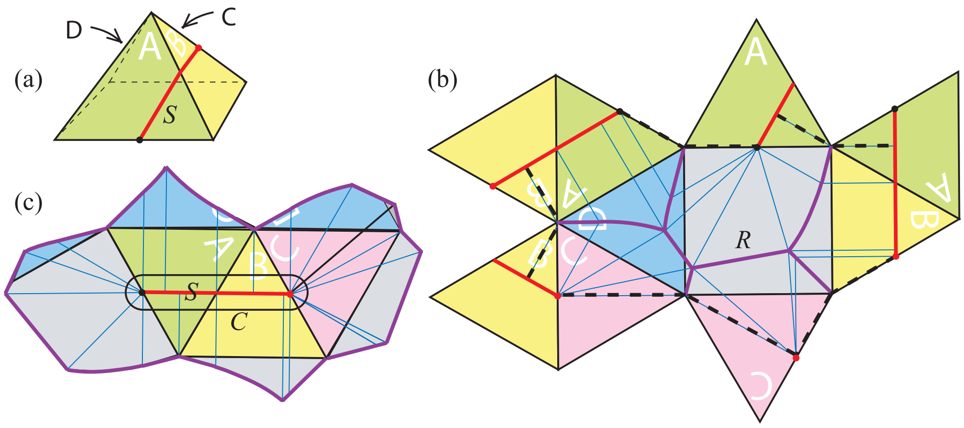

FIGURE 1. Source unfolding from an open geodesic: (a) a pyramid with an open geodesic curve S crossing two faces; (b) the ridge tree R lies in two faces, the base and face D. (The dashed lines, together with segments of S delimit a "dual" unfolding where the paths are attached to R.); (c) the source unfolding showing paths emanating radially from the open geodesic, and showing the convex curve C relevant to sun unfolding.

Our main result is a generalized unfolding, called a *sun unfolding*, that preserves the property that shortest paths emanate in a radially monotone way, although they no longer radiate from a point. We begin with an easy generalization where the point is

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replaced by a curve S that unfolds to a straight line segment (an *open geodesic curve*). Cutting at the ridge tree of points that have more than one shortest path to S produces an unfolding in which the shortest paths from S radiate from the unfolded S; see Figure 1. (All proofs can be found in the long version of the paper.)

For our general sun unfolding, the paths emanate radially, not from a point or a segment, but from a tree S, and the paths are not necessarily shortest paths from S. We define both S and the paths based on a convex curve C on the surface of the polyhedron. Let S be the ridge tree of C on the convex side and let R be the ridge tree of C on the other side. Let $\mathcal{G}$ be the set of all shortest paths to C, where we glue together any paths that reach the same point of C from opposite sides. We prove that the paths emanate in a *radially monotone way* from the unfolded S, and hence that the polyhedron unfolds into a non-overlapping planar surface if we make the following cuts: cut R and, for every vertex v on the convex side of C, cut a shortest path from v to C and continue the cut across C, following a geodesic path, until reaching R. See Figure 2.

Our result generalizes source unfolding from a point or an open geodesic, by taking C to be the locus of points at distance ε from the source. See the curve C in Figure 1(c). Our result is related to recent work of Itoh, O'Rourke, and Vîlcu on "star unfolding via a quasigeodesic loop" [**4**]. A quasigeodesic loop is a special case of a convex polygonal curve. Itoh et al. cut the polyhedron at the loop, unfold both halves (keeping R and S intact) and attach the resulting two pieces. Their unfolding of the convex side is the same one that we use. See Figure 3. Itoh, O'Rourke, and Vîlcu also have an interesting alternative unfolding where the convex curve C remains connected (developing as a path) while S and R are cut [**3, 5**]. This is possible only for special convex polygonal curves.

1 Sun unfolding

We define *sun unfolding* of a convex polyhedron P relative to a closed convex curve C on P. For the purpose of this note, we consider only curves composed of a finite number of line segments and circular arcs. In the long version of the paper we discuss more general convex curves. The curve C splits P into two "halves", the convex or *interior* side C_I, and the *exterior* side C_E. For a point c on C (notated $c \in C$), let $\alpha_I(c)$ be the surface angle of C_I between the left and right tangents at c, and let $\alpha_E(c)$ be the surface angle of C_E between those tangents. Then $\alpha_I(c) + \alpha_E(c) \leq 2\pi$, with equality unless c is a vertex of P. Also, $\alpha_I(c) \leq \pi$. A point c with $\alpha_I(c) < \pi$ is called an *internal corner* of C. A point $c \in C$ with $\alpha_E(c) < \pi$ is called an *external convex corner* of C.

The *ridge tree* (a.k.a "cut locus") in C_I [or C_E] is the closure of the set of points that have more than one shortest path to C. Let S be the ridge tree of C in C_I, and let R be the ridge tree in C_E. Among all the shortest paths from points of C_I to C, let $\mathcal{G}_I$ be the maximal ones. Among all the shortest paths from points of C_E to C, let $\mathcal{G}_E$ be the maximal ones. If $c \in C$ has $\alpha_I(c) = \alpha_E(c) = \pi$, then we concatenate together the unique paths of $\mathcal{G}_I$ and $\mathcal{G}_E$ that are incident to c. Let $\mathcal{G}$ be the resulting set of paths, together with any leftover paths of $\mathcal{G}_I$ and any leftover paths of $\mathcal{G}_E$.

Lemma 1.1 *Both R and S are trees. Every vertex of P lies in R or S (or both). Every internal corner of C is a leaf of S. Every external convex corner of C is a leaf of R. Every path of $\mathcal{G}$ goes from S to R and includes a point of C. The surface of P is covered by S, R, and $\mathcal{G}$. Furthermore, any point not on S or R is in a unique path of $\mathcal{G}$.*

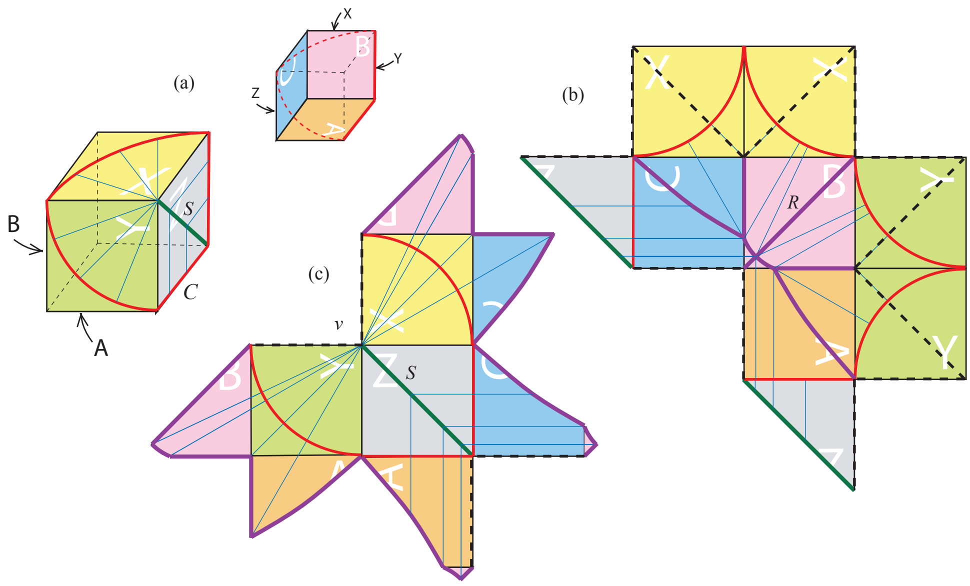

FIGURE 2. Sun unfolding with respect to a convex curve: (a) a cube with a convex curve C and the ridge tree S on the convex side; (b) the ridge tree R on the non-convex side of C. (The dashed lines delimit a "dual" unfolding where the paths are attached to R.); (c) the sun unfolding showing paths emanating radially from S. Note that the two vertices on S require cuts $\gamma(\cdot)$ to R, shown with dashed lines.

Let v be a vertex of P. If v is not in R, then it is in S, and we let $\gamma(v)$ be a path of $\mathcal{G}$ incident to v. The choice of $\gamma(v)$ is not unique in general, but we fix one $\gamma(v)$. Observe that each $\gamma(v)$ is a path from v to R, consisting of a shortest path from v to C possibly continued geodesically to R. We define *sun cuts* with respect to C to consist of R and the paths $\gamma(v)$, for v a vertex of P in $C_I \cup C$. Note that a vertex on C may be a leaf of R, in which case $\gamma(v)$ has length 0.

Theorem 1.2 *Let C be a closed convex curve on the surface of a convex polyhedron P, such that C is composed of a finite number of line segments and circular arcs. Then sun cuts with respect to C unfold the surface of P into the plane without overlap.*

To prove the theorem, we first show that the sun cuts form a tree that reaches all vertices of P—hence the surface unfolds to the plane. To show that the unfolded surface does not overlap, we prove by shrinking C and applying induction that S unfolds without overlap and that the paths of $\mathcal{G}$ emanate from the unfolded S in a *radially monotone way*, defined as follows. Make a tour clockwise around the unfolded S, travelling in the plane an infinitesimal distance away from the unfolded S. See Figure 3(b). Parts of the tour are off the unfolded surface P. In particular, whenever a cut reaches S, there will be a gap in the unfolding. Apart from the gaps, at any point of the tour we are at a point p of $P - (S \cup R)$, so by Lemma 1.1 there is a unique path $\gamma(p) \in \mathcal{G}$ containing p and extending to R. Extend this path to a ray, and let $f(p)$ be the corresponding point on the circle at infinity. We say that $\mathcal{G}$ *emanates from the unfolded S in a radially monotone way* if, as p tours clockwise around S, $f(p)$ progresses clockwise around the circle at infinity, i.e., if $p'' > p' > p$ along the tour then $f(p'') \geq f(p') \geq f(p)$ clockwise around the circle at infinity.

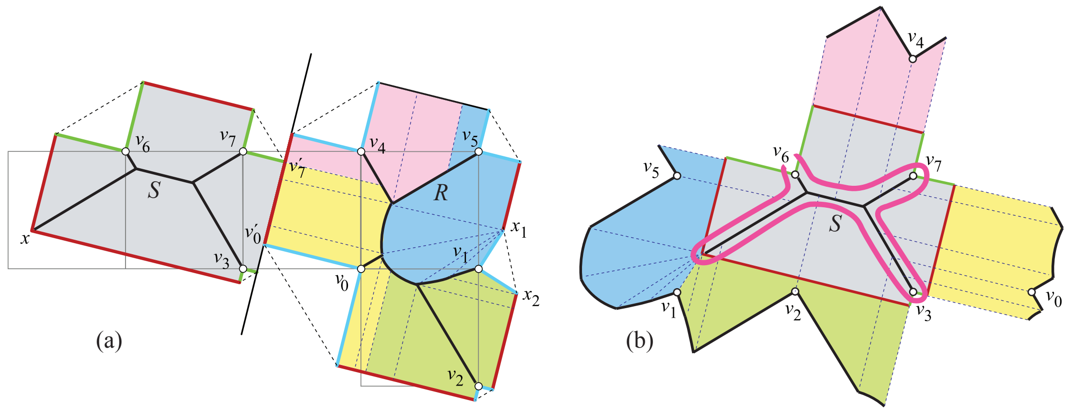

FIGURE 3. The sun unfolding with respect to a geodesic loop Q on a cube, based on an example from Fig. 1 of Itoh et al. [**4**]: (a) the ridge trees S and R on the two sides of the curve Q (superimposed on the unfolding from [**4**]); (b) the sun unfolding with respect to Q, showing a tour around S and the paths emanating in a radially monotone way from S. The faint horizontal and vertical lines in (a) are the edges of the cube.

2 Conclusion

Our sun unfolding generalizes one of the basic unfolding methods for convex polyhedra, namely the source unfolding from a point. The other basic unfolding method is the *star unfolding* from a point s, where the polyhedron surface is cut along a shortest path from every vertex to s [**1**]. This is dual to the source unfolding in that the shortest paths are attached in one case to s (for source unfolding) and in the other case to the ridge tree (for star unfolding). The dual of our sun unfolding would be to attach the paths of $\mathcal{G}$ to the ridge tree R and cut the ridge tree S and paths of $\mathcal{G}$ from vertices to S. See for example Figure 1(b) and Figure 2(b). We conjecture that this unfolds without overlap. A first step would be to prove that the star unfolding from an open geodesic unfolds without overlap.

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