Polyhedral Characterization of Reversible Hinged Dissections

Jin Akiyama* Erik D. Demaine† Stefan Langerman‡

Abstract

We prove that two polygons A and B have a reversible hinged dissection (a chain hinged dissection that reverses inside and outside boundaries when folding between A and B) if and only if A and B are two noncrossing nets of a common polyhedron. Furthermore, *monotone* reversible hinged dissections (where all hinges rotate in the same direction when changing from A to B) correspond exactly to noncrossing nets of a common convex polyhedron. By envelope/parcel magic, it becomes easy to design many hinged dissections.

1 Introduction

Given two polygons A and B of equal area, a *dissection* [9] is a decomposition of A into pieces that can be re-assembled (by translation and rotation) to form B. In a (chain) *hinged* dissection [10], the pieces are hinged together at their corners to form a chain, which can fold into both A and B, while maintaining connectivity between pieces at the hinge points. Figure 1 shows a famous example by Dudeney [8]. Many known hinged dissections (including Figure 1) are *reversible* (originally called *Dudeney dissection* [3]), meaning that the outside boundary of A goes inside of B after the reconfiguration, while the portion of the boundaries of the dissection inside of A become the exterior boundary of B. In particular, the hinges must all be on the boundary of both A and B, in the opposite counterclockwise order. We thus view reversible hinged dissections as a *cyclic* chain of pieces and hinges, because the choice of the hinge to cut to perform the reconfiguration is irrelevant: the two endpoints of the chain meet in both configurations. Other papers [4, 2] call the pair A, B of polygons (instead of the hinged dissection) *reversible*.

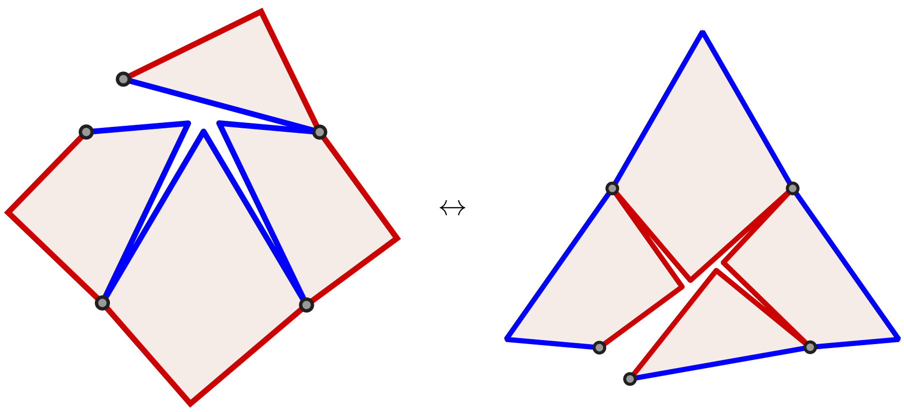


Figure 1: Dudeney's reversible hinged dissection of a square to an equilateral triangle [8].

Without the reversibility restriction, Abbott et al. [1] showed that any two polygons of same area have a hinged dissection. Properties of reversible pairs of polygons were studied by Akiyama

*Tokyo University of Science
†CSAIL, Massachusetts Institute of Technology
‡Directeur de Recherches du F.R.S-FNRS, Université Libre de Bruxelles

et al. [3, 4]. A recent paper [2] described the *parcel magic* method to generate reversible hinged dissections. This method works by cutting open, unfolding, and flattening a polyhedron in two different ways such that the cut trees of the two unfoldings do not intersect. The special case when the polyhedron is a dihedron (flat doubly covered polygon) is called *envelope magic*. The purpose of this paper is to formalize this method and show that this characterization is in some sense complete, that is, that every reversible hinged dissection can be constructed this way.

More precisely, we show the following three results:

1. Two polygons A, B have a reversible hinged dissection if and only if A and B are two noncrossing nets of a common polyhedron (Theorem 3.1).

2. Two polygons A, B have a *monotone* reversible hinged dissection (where all the turn angles of all hinges increase from A to B) if and only if A and B are two noncrossing nets of a common *convex* polyhedron (Theorem 4.1).

3. Two polygons A, B have a *nondegenerate* reversible hinged dissection (where each hinge touches just its two incident pieces), if and only if A and B are two noncrossing nets of a common *convex* polyhedron that have only one cut incident to each polyhedron vertex (Theorem 5.2).

2 Noncrossing Nets

The heart of our results is a lemma about circumnavigating a polyhedron between two noncrossing cut trees of unfoldings.

First we need some terminology. In this paper, a *polyhedron* is always homeomorphic to a sphere. An *unfolding* of a polyhedron P cuts the surface of P using a *cut tree* T,[1] spanning all vertices of P, such that the cut surface $P \setminus T$ can be unfolded into the plane without overlap by opening all dihedral angles between the (possibly cut) faces. The planar polygon that results from this unfolding is called a *net* of P. Two trees T_1 and T_2 drawn on a surface are *noncrossing* if pairs of edges of T_1 and T_2 intersect only at common endpoints and, for any vertex v of both T_1 and T_2, the edges of T_1 (respectively, T_2) incident to v are contiguous in clockwise order around v. Two nets of a common polyhedron are noncrossing if their cut trees are noncrossing.

Lemma 2.1. *Let T_1, T_2 be noncrossing trees drawn on a polyhedron P, each of which spans all vertices of P. Then there is a cycle C passing through all vertices of P such that C separates the edges of T_1 from edges of T_2, i.e., the (closed) interior (yellow region, see Figure 2) of C includes all edges of T_1 and the (closed) exterior of C includes all edges of T_2. Furthermore, all such cycles visit the vertices of P in the same order.*

Proof. Refer to Figure 3. Let G be the union of T_1 and T_2. Because T_1 and T_2 are noncrossing, G is a planar graph. Let $\alpha > 0$ be a third of the smallest angle between any two incident edges in G, or $90°$, whichever is smaller. Let $\varepsilon > 0$ be a third of the smallest distance[2] between any edge of G and a vertex not incident to that edge. View each edge of G as the union of two directed half-edges. For every half-edge u, v in G, its *sidewalk* is a polygonal path u, p_{uv}, q_{uv}, v composed of three segments such that

1. the counterclockwise angles $\angle p_{uv}, u, v$ and $u, v, \angle q_{uv}$ are both α (placing p_{uv} and q_{uv} on the left of the directed line u, v); and

2. both p_{uv} and q_{uv} are at distance ε from the segment u, v.

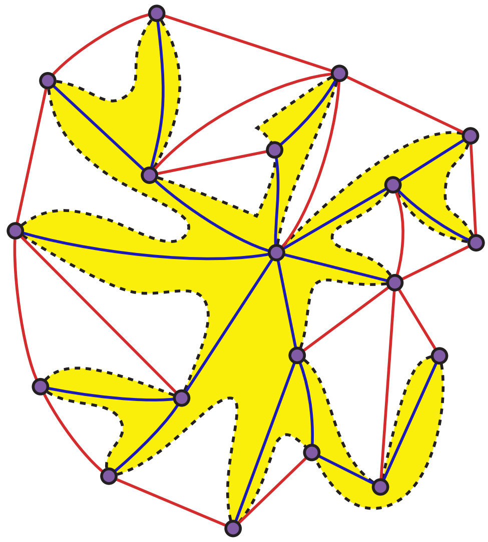
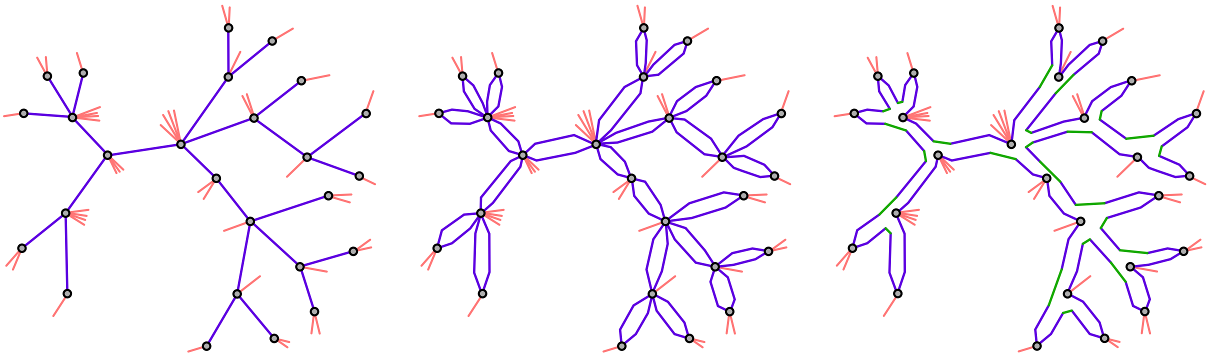


Figure 2: Example of Lemma 2.1. The edges of T_1, T_2 are colored blue, red, respectively.



(a) Tree T_1 (purple) and its local interaction with tree T_2 (red)

(b) Sidewalks

(c) Crosswalks

Figure 3: Proof of Lemma 2.1.

By construction, u, v is the unique closest edge of G from any point on its sidewalk. Thus, no two distinct sidewalks intersect and sidewalks do not intersect edges of G.

Construct an Euler tour of T_1 (Figure 3a) that is noncrossing and traverses clockwise around T_1, and replace each step from u to v in the tour by the sidewalk of u, v. The concatenation of all these sidewalks forms a clockwise cycle that visits each vertex v as many times as the degree of v (Figure 3b). For any two consecutive sidewalks $u, p_{uv}, q_{uv}, v, p_{vw}, q_{vw}, w$ where the wedge u, v, w does not contain an edge of T_2 incident to v, shorten the walk by using the *crosswalk* q_{uv}, p_{vw} to obtain $\ldots, p_{uv}, q_{uv}, p_{vw}, q_{vw}, \ldots$, thereby avoiding a duplicate visit of v. Because T_1 and T_2 are noncrossing, all but one of the visits of each vertex v will be removed by using crosswalks (Figure 3c).

The resulting walk is a simple closed Jordan curve C that visits each vertex of G exactly once. Because C does not intersect T_1 and T_2, and locally separates the edges of T_1 and T_2 at each vertex, and because T_1 and T_2 are connected, the curve C separates P into two regions, one containing T_1 and the other containing T_2.

Finally, we show that the order of vertices of P visited by any such cycle C is determined by P, T_1, and T_2. Consider the planar graph $T_1 \cup T_2$ drawn on P. We claim that every face of

[1]For simplicity, we assume that the edges of T are drawn using segments along the surface of P, and that vertices of degree 2 can be used in T to draw any polygonal path.

[2]We here refer to the geodesic distance on the surface of P.

this graph consists of at most one path of edges from T_1 and at most one path of edges from T_2. Otherwise, we would have at least two components of T_1 and at least two components of T_2, neither of which could be connected interior to the face (because the face is empty), and at most one of which could be connected exterior to the face (by planarity and the noncrossing property), contradicting that T_1 and T_2 are both trees. Therefore, every face with at least one edge from T_1 and at least one edge from T_2 locally forces where C must go, connecting the two vertices with incident face edges from both T_1 and T_2. Every vertex of P has at least one incident edge from each of the spanning trees T_1 and T_2, so has two incident such faces. In this way, we obtain the forced vertex ordering of C. □

3 Reversible Hinged Dissections

We can now give our first characterization.

Theorem 3.1. *Two polygons A and B have a reversible hinged dissection if and only if A and B are two noncrossing nets of a common polyhedron.*

Proof. To prove one direction ("only if"), it suffices to glue both sides of the pieces of the dissection as they are glued in both A and B to obtain a polyhedral metric homeomorphic to a sphere. A result of Burago and Zalgaller [14, 15][3] shows that any such metric corresponds to the surface of some (not necessarily unique, possibly nonconvex) polyhedron [12]. In the other direction ("if"), we use Lemma 2.1 to define the sequence of hinges. Now the cut tree T_B of net B is completely contained in the net A and determines the hinged dissection. □

4 Monotone Reversible Hinged Dissections

Often times, reversible hinged dissections are also *monotone* meaning that, if we order the hinges in the dissection counterclockwise around the boundary of A, then the turn angle at every hinge decreases (i.e., turns to the right) when transforming from A to B. (This definition is symmetric in A and B because B's counterclockwise order of the hinges is the reverse of A's counterclockwise order of the hinges.) Recall that we view reversible hinged dissections as a cyclic chain, so the monotonicity definition also measures the change in angle of the hinge that is cut open in order to perform the reconfiguration but which recombines into a vertex in both the A and B configurations. Figure 1 is monotone, while Figure 4 shows a hinged dissection that is reversible but not monotone.

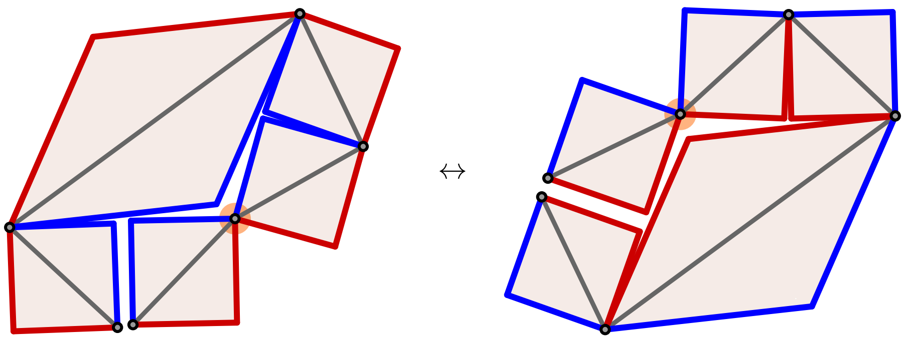


Figure 4: Reversible hinged dissection that is not monotone (nor is it nondegenerate), because of the highlighted vertex.

[3]Burago and Zalgaller first proved the result for any orientable surface [14] but later noted and fixed a flaw in their construction [15]. At the same time, they generalized their result to a stronger statement about possibly non-orientable surfaces.

Our second characterization shows that monotonicity in the hinged dissection is equivalent to convexity of the polyhedron:

Theorem 4.1. *Two polygons A and B have a monotone reversible hinged dissection if and only if A and B are two noncrossing nets of a common convex polyhedron.*

Proof. Let v be a hinge of the monotone reversible hinged dissection. Pick two reference points v^- and v^+ in the neighborhood of v and in the pieces before and after hinge v, respectively, in counterclockwise order around the boundary of A. Let α_v be the counterclockwise angle $\angle v^-, v, v^+$ when the dissection forms polygon A, and let α'_v be the same angle when the dissection forms polygon B. Because the dissection is monotone, $\alpha'_v \geq \alpha_v$ for all v.

Just as in Theorem 3.1, glue both sides of the dissection as they are glued in both A and B to obtain a polyhedral metric homeomorphic to a sphere. Observe now that the total angle glued at vertex v is exactly $\alpha_v + (2\pi - \alpha'_v) \leq 2\pi$. By Alexandrov's Theorem [5, 7, Section 23.3], there exists a unique convex polyhedron (up to rigid transformations) whose surface has this intrinsic metric.

In the other direction, suppose we have two noncrossing nets of a convex polyhedron P. Use Lemma 2.1 to find a cycle C separating T_A and T_B on the surface of P and to define the sequence of hinges, and cut both trees to obtain the dissection. Pick points v^- and v^+ before and after v on C and in the neighborhood of v. Let α_v be the angle $\angle v^-, v, v^+$ in net A and on the surface of P, and β_v be the angle $\angle v^+, v, v^-$ in net B and on the surface of P. Because P is convex, $\alpha_v + \beta_v \leq 2\pi$. The angle $\alpha'_v = \angle v^-, v, v^+$ when the dissection forms polygon B is exactly $2\pi - \beta_v \geq \alpha_v$ for every hinge v, and so the dissection is monotone. □

Two unfoldings of a common convex polyhedron is in some sense the dual notion of two convex polyhedra with a *common unfolding*, a topic that has been studied extensively; see [13, 7, Section 25.8.3].

5 Nondegenerate Reversible Hinged Dissections

An interesting special case of a monotone reversible hinged dissection is when every hinge touches only its two incident pieces in both its A and B configurations, and thus A and B are the only possible such configurations. We call these *nondegenerate* reversible hinged dissections. (For example, Figure 1 is nondegenerate, and Figure 4 is degenerate and nonmonotone, while Figure 5 is degenerate and monotone.)

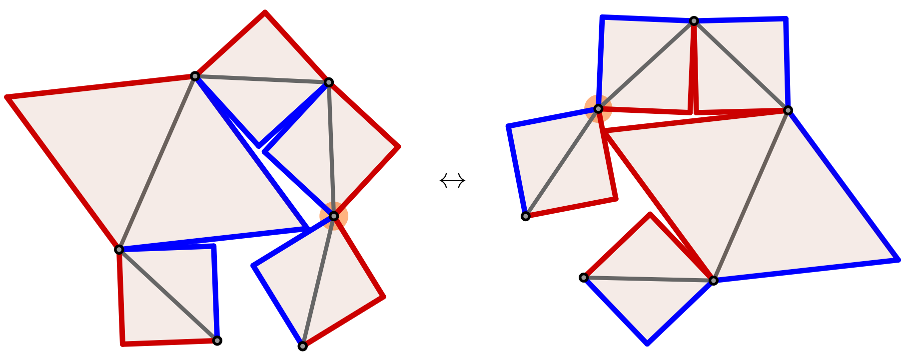


Figure 5: Reversible hinged dissection that is monotone but degenerate, because of the highlighted vertex.

Lemma 5.1. *Every nondegenerate reversible hinged dissection is strictly monotone, i.e., monotone and all turn angles change between A and B.*

Proof. Pick the reference points v^- and v^+ and define angle α_v and α'_v as in Theorem 4.1. Because the dissection is nondegenerate, the two pieces attached to hinge v touch on the inside of A. Therefore, for any hinge angle $\angle v^-, v, v^+$ less than α_v, those two pieces would intersect. Because no two piece intersect when the dissection forms polygon B, $\alpha'_v \geq \alpha_v$ for all v and the dissection is monotone. Furthermore, we cannot have $\alpha'_v = \alpha_v$, because then the two pieces would touch on both sides of the hinge, meaning that it is not a hinge at all. □

The distinguishing feature of nondegenerate reversible hinged dissections is that their corresponding unfoldings are *nondegenerate* in the sense that every vertex of the polyhedron has just a single incident cut (degree 1) in the cut tree.

Theorem 5.2. *Two polygons A and B have a nondegenerate reversible hinged dissection if and only if A and B are two nondegenerate noncrossing nets of a common convex polyhedron.*

Proof. By Lemma 5.1, a nondegenerate reversible hinged dissection between A and B is strictly monotone, so by Theorem 4.1, A and B are two noncrossing nets of a common convex polyhedron. When gluing the pieces along both dissection boundaries to form the convex polyhedral metric, we form vertices exactly at the hinges (by strict monotonicity). By nondegeneracy, no other piece touches a hinge, so the resulting polyhedron vertex has only one cut in each of the two unfoldings (corresponding to the opened edge in each hinged dissection).

In the other direction, if A and B are two nondegenerate noncrossing nets of a common convex polyhedron, then by Theorem 4.1, they have a monotone reversible hinged dissection between A and B. Each of the two states of the hinged dissection is formed by cutting the cut tree of one unfolding (e.g., A) while leaving attached the cut tree of the other unfolding (e.g., B). Because both unfoldings are nondegenerate, at each hinge, the pieces share one side (e.g., B) while leaving an empty sector angle on the other side (e.g., A), so no other piece can be incident to the hinge. Therefore the unfolding is nondegenerate. □

Figure 6 shows two examples of hinged dissections resulting from these techniques. Historically, many hinged dissections (e.g., in [9, 10]) have been designed by overlaying tessellations of the plane by shapes A and B. This connection to tiling is formalized by the results of this paper, combined with the characterization of shapes that tile the plane isohedrally as unfoldings of certain convex polyhedra [11].

References

[1] Timothy G. Abbott, Zachary Abel, David Charlton, Erik D. Demaine, Martin L. Demaine, and Scott Duke Kominers. Hinged dissections exist. *Discrete & Computational Geometry*, 47(1):150–186, 2012.

[2] Jin Akiyama, Stefan Langerman, and Kiyoko Matsunaga. Reversible nets of polyhedra. In *Revised Papers from the 18th Japan Conference on Discrete and Computational Geometry and Graphs*, pages 13–23, Kyoto, Japan, September 2015.

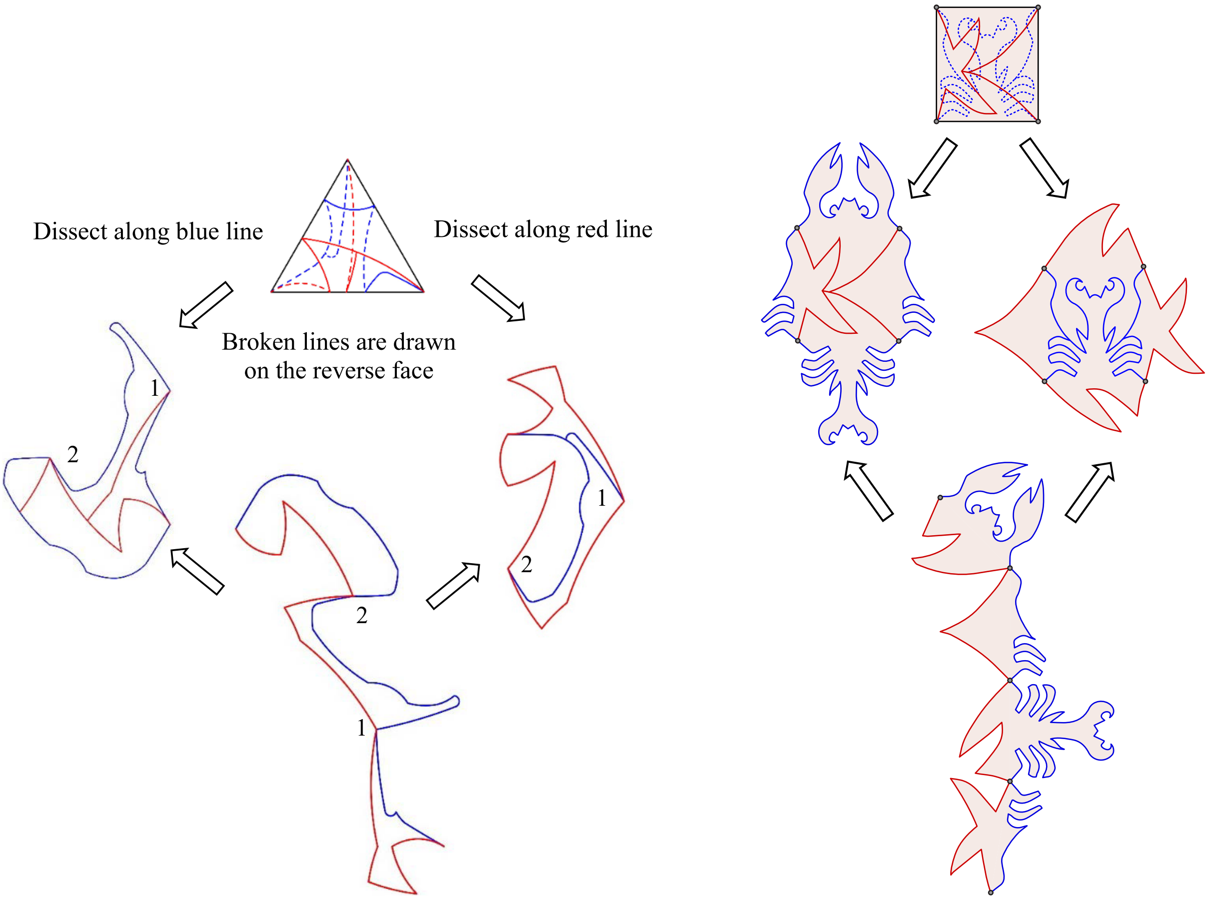
[3] Jin Akiyama and Gisaku Nakamura. Dudeney dissection of polygons. In *Revised Papers from the Japan Conference on Discrete and Computational Geometry*, volume 1763 of *Lecture Notes in Computer Science*, pages 14–29, Tokyo, Japan, December 1998.

[4] Jin Akiyama and Hyunwoo Seong. An algorithm for determining whether a pair of polygons is reversible. In *Proceedings of the 3rd Joint International Conference on Frontiers in Algorithmics and Algorithmic Aspects in Information and Management*, pages 2–3, Dalian, China, June 2013.

[5] A. D. Alexandrov. *Convex Polyhedra.* Springer, 2005. Translation of Russian edition, 1950.

[6] Yu. D. Burago and V. A. Zalgaller. Isometric piecewise linear immersions of two-dimensional manifolds with polyhedral metrics in $\mathbb{R}^3$. *St. Petersburg Math. J.*, 7:369–385, 1996. Translation by S.V. Ivanov.

[7] Erik D. Demaine and Joseph O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra.* Cambridge University Press, July 2007.



(a) Two noncrossing nets of a doubly covered triangle.

(b) Lobster to fish: two noncrossing nets of a doubly covered rectangle.

Figure 6: Two nondegenerate reversible hinged dissections found by parcel and envelope magic.

[8] Henry E. Dudeney. Puzzles and prizes. *Weekly Dispatch*, 1902. The puzzle appeared in the April 6 issue of this column. A discussion followed on April 20, and the solution appeared on May 4.

[9] Greg N. Frederickson. *Dissections: Plane and Fancy*. Cambridge University Press, November 1997.

[10] Greg N. Frederickson. *Hinged Dissections: Swinging & Twisting*. Cambridge University Press, August 2002.

[11] Stefan Langerman and Andrew Winslow. A complete classification of tile-makers. In *Abstracts from the 18th Japan Conference on Discrete and Computational Geometry and Graphs*, Kyoto, Japan, September 2015.

[12] Joseph O'Rourke. On folding a polygon to a polyhedron. arXiv:1007.3181, July 2010. http://arXiv.org/abs/1007.3181.

[13] Ryuhei Uehara. A survey and recent results about common developments of two or more boxes. In *Origami^6: Proceedings of the 6th International Meeting on Origami in Science, Mathematics and Education*, volume 1 (Mathematics), pages 77–84. American Mathematical Society, 2014.

[14] Ю. Д. Бураго and В. А. Залгаллер. Реализация разверток в виде многогранников (polyhedral embedding of a net). *Вестник ЛГУ*, 15:66–80, 1960. In Russian with English summary.

[15] Ю. Д. Бураго and В. А. Залгаллер. Изометрические кусочно-линейные погружения двумерных многообразий с полиэдральной метрикой в $\mathbb{R}^3$. *Алгебра и анализ*, 7:76–95, 1995. In Russian, translation in [6].