Relaxed Gabriel Graphs

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Abstract

We study a new family of geometric graphs that interpolate between the Delaunay triangulation and the Gabriel graph. These graphs share many properties with β-skeletons for $\beta \in [0, 1]$ (such as sublinear spanning ratio) with the added benefit of planarity (and consequently linear size and local routability).

1 Introduction

A *geometric graph* is a finite graph whose vertices are points in the plane and whose edges are represented by straight line segments between their endpoints. We consider here a class of geometric graphs called *proximity graphs* [10]. Two points p and q in the vertex set S are deemed "close" to each other if some neighborhood of the segment pq is empty of other points in S. The corresponding proximity graph contains an edge between every such close pair of points. In this paper we study three well-known families of proximity graphs: Gabriel graphs, Delaunay triangulations, and β-skeletons and combine the interesting aspects of each.

The Gabriel graph [9] is a proximity graph where two vertices p and q are joined by an edge if and only if the disk with diameter pq has no other points of S in its interior. If the empty circle instead merely has to pass through p and q, and not necessarily have pq as its diameter, then the resulting graph is the Delaunay triangulation. The Delaunay triangulation was introduced by Delaunay in 1934 [6] and has been studied extensively to this day. (See [13] for a survey.) Finally, the β-skeleton is a well-known proximity graph where the shape and size of the region that needs to be empty in order for two vertices of the graph to be connected by an edge depends on a parameter β. For $\beta = 1$, the β-skeleton of S is the Gabriel graph; as β decreases, more and more edges are added to the β-skeleton. In general, β-skeletons are not planar for values of $\beta < 1$, and for small enough β they can have $\Theta(n^2)$ edges.

Let $d_2(p, q)$ denote the Euclidean distance between points p and q, and $d_G(p, q)$ the Euclidean length of the shortest path between p and q in the graph G. A graph $G = (V, E)$ in the plane is a *t-spanner* if

$$\frac{d_G(u, v)}{d_2(u, v)} \leq t \quad \text{for all } u, v \in V,$$

where $t \geq 1$ is called the *spanning ratio* of G.

Intuitively, graphs having a large number of edges are more likely to have a smaller spanning ratio. In particular, the spanning ratio of the complete graph (having an edge between every pair of vertices) is 1. However, in 1986, Chew [4] showed that every point set has a *planar* 2-spanner. He also conjectured that the Delaunay triangulation is an $O(1)$-spanner. Dobkin, Friedman, and Supowit [7] proved this conjecture in the early 90's, establishing an upper bound of $(1 + \sqrt{5})/2 \cdot \pi \approx 5.08$ on the spanning ratio of the Delaunay triangulation. In 1992, Keil and Gutwin [11] decreased the bound to $2\pi/(3 \cos \pi/6) \approx 2.42$. A tight bound is still not known.

The Gabriel graph, on the other hand, has an unbounded spanning ratio [2, 8, 14]. In other words, there exists a family of point sets such that the spanning ratio of the Gabriel graph of each such point set S in the family is a growing function of the size of S. Somewhat surprisingly, an even stronger result holds: for any value of $\beta > 0$, β-skeletons have an unbounded spanning ratio. This result seems counterintuitive because the spanning ratio of a β-skeleton is 1 for $\beta = 0$.

Ideally, we would like to have a family of proximity graphs that have a linear number of edges (like Delaunay and Gabriel graphs), are planar, and are parameterized (similar to β-skeletons) to allow tuning of properties of the graph. To this end, we define a parameterized class of graphs called the *relaxed Gabriel graph* (RGG). The relaxed Gabriel graph of a point set S is the intersection of the Delaunay triangulation and a β-skeleton of S. Depending on the choice of β, the spanning ratio of the relaxed Gabriel graph ranges between that of the Delaunay triangulation and the Gabriel graph.

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We explore the various properties of the relaxed Gabriel graph. In particular, we show in Section 3 that the worst-case spanning ratio is $n^{\Theta(f(\beta))}$ where $0 \leq f(\beta) \leq 1$, like β-skeletons. We also show in Section 4 that relaxed Gabriel graphs admit competitive online routing strategies, in particular by exploiting their planarity. Finally we mention in Section 5 a variation with better spanning ratio.

2 Definitions

In what follows, we assume that no four points are concyclic.

Definition 1 (Delaunay Triangulation) *Given a set S of points in general position, the* Delaunay triangulation $DT(S)$ *of S is the graph whose vertex set is S and that has an edge between two vertices x and y if and only if there exists a closed disk D such that:*

1. *x and y are on the boundary of D, and*
2. *$D \cap S \setminus \{x, y\} = \emptyset$.*

Definition 2 (β-skeleton) *Given a point set S in general position and a real number $\beta \in [0, 1]$, the β-*skeleton $G_\beta(S)$ *is the graph whose vertex set is S and that has an edge between two vertices x and y if and only if, for every point $c \in S \setminus \{x, y\}$, the absolute angle $\angle xcy \in [0, \pi]$ is at most α, where $\alpha = 2 \arcsin \beta$.*

In other words, there exists an edge between two points x and y if the intersection of the two disks of radius $d_2(x, y)/(2\beta)$ having x and y on their boundaries is empty. We call this intersection the *β-region* of xy. When $\beta = 1$, this intersection is exactly the disk with diameter xy. The corresponding graph $G_1(S)$ is the *Gabriel graph*.

The usual definition of β-skeletons extends to $\beta \in [0, \infty]$. In particular, for $\beta \in [1, \infty[$, the corresponding empty region is the intersection of the two disks of radius $\beta\, d_2(x, y)/2$ and centered at the points $(1-\beta/2)x + (\beta/2)y$ and $(\beta/2)x + (1 - \beta/2)y$, respectively (see Figure 1). For $\beta \geq 1$, the β-skeleton is planar (being a subgraph of the Gabriel graph). For $\beta \in [0, 1]$, this may not be the case. This is the range we concentrate on.

The relaxed Gabriel graph for a point set S is parametrized by an angle $\alpha \in [0, \pi]$ and denoted by $\mathrm{RGG}_\alpha(S)$. The formal definition is given below.

Definition 3 (Relaxed Gabriel graph) *Given a point set S in general position and a real number $\alpha \in [0, \pi]$, the* relaxed Gabriel graph $\mathrm{RGG}_\alpha(S)$ *is the graph with vertex set S and that has an edge between two vertices x and y if and only if there exists a closed disk D with center c such that:*

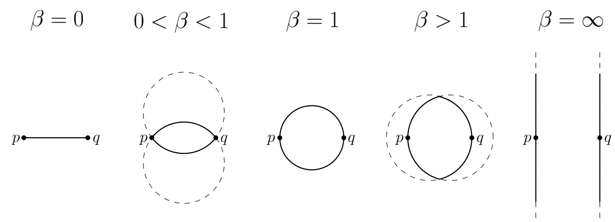

Figure 1: Empty regions defining edges in β-skeletons.

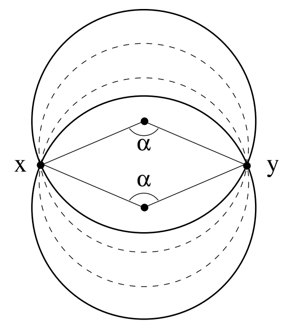

Figure 2: Edge xy exists if the two bold disks or any disk contained in their union and touching x and y is empty of points in S.

1. *x and y are on the boundary of D,*
2. *$D \cap S \setminus \{x, y\} = \emptyset$, and*
3. *the angle at c in the triangle xcy is at least α.*

Figure 2 illustrates this definition. Relaxed Gabriel graphs are planar and connected. They interpolate between the Gabriel graph (for $\alpha = \pi$) and the Delaunay triangulation (for $\alpha = 0$).

Lemma 1 *The graph $\mathrm{RGG}_\alpha(S)$ is the Gabriel graph of S for $\alpha = \pi$, and the Delaunay triangulation of S for $\alpha = 0$. Given two real numbers $\alpha > \alpha'$ with $\alpha, \alpha' \in [0, \pi]$, we have $\mathrm{RGG}_\alpha(S) \subseteq \mathrm{RGG}_{\alpha'}(S)$.*

By definition, the graph $\mathrm{RGG}_\alpha(S)$ is a subgraph of $DT(S)$. Moreover, we show that the set of edges of $\mathrm{RGG}_\alpha(S)$ is exactly the intersection of the edge sets of $DT(S)$ and the β-skeleton $G_\beta(S)$ for $\beta = \sin(\alpha/2)$.

Lemma 2 (Alternative definition)

$$\mathrm{RGG}_\alpha(S) = G_\beta(S) \cap DT(S), \text{ where } \beta = \sin(\alpha/2).$$

Proof. If an edge xy belongs to $\mathrm{RGG}_\alpha(S)$, then there exists an empty disk with x and y on its boundary; thus, the edge belongs to $DT(S)$. The center c of this disk makes an angle of at least α with x and y; hence, the disk also contains the β-region of xy for $\beta = \sin(\alpha/2)$. Therefore, xy also belongs to $G_\beta(S)$.

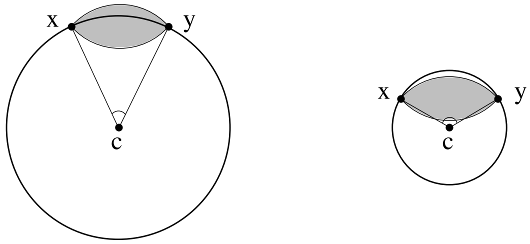

Figure 3: Alternative definition of the relaxed Gabriel graphs.

On the other hand, if there exists an edge $xy \in G_\beta(S) \cap DT(S)$, then there exists an empty disk with x and y on its boundary, and the β-region of xy is empty. If this disk contains the β-region of xy, then by the choice of β, the center c of this disk is such that $\angle xcy \geq \alpha$ (see Figure 3). Otherwise, there exists a smaller empty disk that contains the β-region and has x and y on its boundary. The center c of this disk is such that $\angle xcy \geq \alpha$. Hence, in both cases, the edge xy belongs to $\mathrm{RGG}_\alpha(S)$. □

As already pointed out, the β-skeleton $G_\beta(S)$ is not necessarily planar for $\beta \in [0, 1]$. By selecting only edges of the Delaunay triangulation, we ensure that RGG_α is planar. As we show in the next two sections, the restriction to Delaunay edges does not affect the upper bound on the spanning ratio of the graph, but the planarity allows us to obtain an efficient online routing algorithm for relaxed Gabriel graphs.

3 Spanning Ratio

We first give an upper bound on the spanning ratio of the graph RGG_α. This bound matches the $O(\sqrt{n})$ bound for Gabriel graphs in the case $\alpha = \pi$, and is constant for $\alpha = 0$, matching the upper bound on the spanning ratio of the Delaunay triangulation [11]. Interestingly, the upper bound is the same as the best known upper bound for β-skeletons, with $\beta = \sin(\alpha/2)$. The proof is essentially the same as that by Bose et al. [2]. Their upper bound is constructive. Because the spanning paths they construct only involve Delaunay edges, they are contained in RGG_α. Next we provide more details.

To prove the upper bound, we construct a walk $W(x, y)$ between any pair of points x, y where xy is an edge of $DT(S)$ (note that in a walk, some vertices may be visited multiple times). For any such pair, it is known that either there is an edge between x and y in $\mathrm{RGG}_\alpha(S)$, or there exists a third point $z \in S$ such that the angle $\angle xzy$ is large.

Lemma 3 *Let xy be an edge of $DT(S)$. For $\beta \in [0, 1]$, either xy is an edge of $G_\beta(S)$, or there exists a unique point z such that xz and zy are also edges of $DT(S)$, and z lies in the β-region of xy.*

Corollary 1 *Let xy be an edge of $DT(S)$. For $\alpha \in [0, \pi]$, either xy is an edge of $\mathrm{RGG}_\alpha(S)$, or there exists a unique point z such that xz and zy are also edges of $DT(S)$, and z lies in the β-region of xy, with $\beta = \sin(\alpha/2)$.*

The definition of the walk $W(x, y)$ is as follows:

$$W(x,y) = \begin{cases} xy & \text{if } xy \in \mathrm{RGG}_\alpha(S), \\ W(x,z) \cup W(z,y) & \text{otherwise.} \end{cases}$$

Bose et al. proved that the overall length $|W(x, y)|$ of this walk is within a polynomial factor of the Euclidean distance between x and y.

Lemma 4 (Bose et al. [2]) *If $W(x, y)$ has m edges, then*

$$|W(x,y)| \leq m^\gamma d_2(x,y),$$

where

$$\gamma = \tfrac{1}{2}\left(1 - \log_2(1 + \cos\tfrac{\alpha}{2})\right).$$

Because the Delaunay triangulation is a spanner, we can find a path in $DT(S)$ between any pair of points x and y, the length of which is within a constant factor of $d_2(x, y)$. Hence, the spanning ratio of RGG_α is within a constant factor of n^γ.

Theorem 5 (Spanning ratio – upper bound) *Given a point set S in general position, the spanning ratio of $\mathrm{RGG}_\alpha(S)$ is $O(n^\gamma)$, where*

$$\gamma = \tfrac{1}{2} - \tfrac{1}{2}\log_2(1 + \cos\tfrac{\alpha}{2}).$$

Because the relaxed Gabriel graph of a point set is a subgraph of the β-skeleton, any lower bound on the spanning ratio of the β-skeleton is also a lower bound on the spanning ratio of the relaxed Gabriel graph. The result of Wang et al. [14] thus implies the following.

Theorem 6 (Spanning ratio – lower bound) *There exists a point set S such that the spanning ratio of $\mathrm{RGG}_\alpha(S)$ is $\Omega(n^\gamma)$, where*

$$\gamma = \tfrac{1}{2} - \tfrac{1}{2}\log\left(1 + \sqrt{\frac{1 + \cos\frac{\alpha}{2}}{2}}\right).$$

4 Routing

One advantage we obtain by defining the relaxed Gabriel graph as a planar subgraph of the β-skeleton is that competitive online routing becomes possible. In contrast, no deterministic online routing algorithm, competitive or not, is known for β-skeletons.

Suppose that we want to route a message from a point s to a point t in $\mathrm{RGG}_\alpha(S)$. Let $F(s,t)$ be the subgraph of $\mathrm{RGG}_\alpha(S)$ spanned by all edges on the boundary of the faces intersected by the line segment st. Then we have the following:

Lemma 7 *There exists a path between s and t in $F(s,t)$, of length at most $O(n^\gamma) \cdot d_2(s,t)$.*

Proof. First suppose that $st \in DT(S)$. Then the walk defined by $W(s,t)$ uses only edges of $F(s,t)$. Otherwise, if $st \notin DT(S)$, there exists a path between s and t in $DT(S)$ that only uses edges of the faces of $DT(S)$ intersected by the segment st. We can replace every edge uv in this path by the walk $W(u,v)$ (possibly the edge uv itself), while still using only edges of $F(s,t)$. Hence in both cases this path exists. $\square$

Bose and Morin [3] proved the same result for the Delaunay triangulation and showed that this condition is sufficient to obtain a $9t$-competitive routing algorithm for $DT(S)$, where $t \approx 5$ is the upper bound on the spanning ratio of the Delaunay triangulation shown by Dobkin et al. [7]. The key observation is that the shortest path from s to t must visit every degree-3 vertex in the shortest path tree in $F(s,t)$ rooted at s. Because these vertices can be recognized locally, the doubling search technique of Baeza-Yates et al. [1] can be used to obtain the following result.

Theorem 8 *There exists an $O(1)$-memory deterministic online routing algorithm capable of routing a message between any two nodes s and t in $\mathrm{RGG}_\alpha(S)$ while travelling distance at most $O(n^\gamma \cdot d_2(s,t))$, where γ is defined as in Theorem 5.*

5 Generalization to Arbitrary Triangulations

Relaxed Gabriel graphs can be obtained by filtering edges in the Delaunay triangulations. The same technique can be applied to other triangulations, where edges that do not belong to the β-skeleton for a chosen value of β in $[0,1]$ are removed. If we start with triangulations having a small guaranteed spanning ratio (such as Chew's variant of Delaunay triangulations based on triangles instead of disks [5]), we can guarantee a better upper bound on the spanning ratio of the resulting pruned graph.

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References

[1] R. Baeza-Yates, J. Culberson, and G. Rawlings. Searching in the plane. *Information and Computation*, 106(2):234–252, 1993.

[2] P. Bose, L. Devroye, W. Evans, and D. Kirkpatrick. On the spanning ratio of gabriel graphs and β-skeleton. *SIAM Journal on Discrete Mathematics*, 20(2):412–427, 2006.

[3] P. Bose and P. Morin. Competitive online routing in geometric graphs. *Theoretical Computer Science*, 324(2-3):273–288, 2004.

[4] L. Chew. There is a planar graph almost as good as the complete graph. In *Proceedings of the 2nd ACM Symposium on Computational Geometry*, pages 169–177, 1986.

[5] L. Chew. There are planar graphs almost as good as the complete graph. *J. Comput. Syst. Sci.*, 39(2):205–219, 1989.

[6] B. N. Delaunay. Sur la sphere vide. *Bull. Acad. Sci. USSR VII: Class. Sci. Math.*, pages 793–800, 1934.

[7] D. Dobkin, S. Friedman, and K. Supowit. Delaunay graphs are almost as good as complete graphs. *Discrete and Computational Geometry*, 5:399–407, 1990.

[8] D. Eppstein. Beta-skeletons have unbounded dilation. *Computational Geometry: Theory and Applications*, 23(1):43–52, 2002.

[9] K. Gabriel and R. Sokal. A new statistical approach to geographic variation analysis. *Systematic Zoology*, 18:259–278, 1969.

[10] J. Jaromczyk and G. Toussaint. Relative neighborhood graphs and their relatives. *Proceedings of the IEEE*, 80(9):1502–1571, 1992.

[11] J. M. Keil and C. Gutwin. Classes of graphs which approximate the complete euclidean graph. *Discrete and Computational Geometry*, 7(1):13–28, 1992.

[12] D. Kirkpatrick and J. Radke. A framework for computational morphology. *Computational Geometry*, pages 217–248, 1985.

[13] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Concepts and Applications of Voronoi Diagrams.* Wiley, 2000. second edition.

[14] W. Wang, X. Li, K. Moaveninejad, Y. Wang, and W. Song. The spanning ratio of beta-skeletons. In *Proceedings of the Canadian Conference on Computational Geometry (CCCG)*, 2003.