Extended Abstract
The 23rd Thailand-Japan Conference on
Discrete and Computational Geometry, Graphs, and Games
Department of Mathematics, Chiang Mai University

Complexity of Simple Folding Orthogonal Crease Patterns

Josh Brunner[1], Erik D. Demaine[1], Dylan Hendrickson[1], Victor Luo[1], and Andy Tockman[1]

[1]MIT Computer Science and Artificial Intelligence Laboratory, 32 Vassar Street, Cambridge, MA 02139, USA
`brunnerj@mit.edu` (J. Brunner), `edemaine@mit.edu` (E. Demaine), `dylanhen@mit.edu` (D. Hendrickson), `vluo@mit.edu` (V. Luo), `tockman@mit.edu` (A. Tockman)

Abstract

Continuing results from JCDCGGG 2016 and 2017, we solve several new cases of the *simple folding problem* — deciding which crease patterns can be folded flat by a sequence of (some model of) simple folds. We give new efficient algorithms for *mixed* crease patterns, where some creases are assigned mountain/valley while others are unassigned, for all 1D cases and for 2D rectangular paper with one-layer simple folds. By contrast, we show strong NP-completeness for mixed crease patterns on 2D rectangular paper with some-layers simple folds, complementing a previous result for all-layers simple folds. We also prove strong NP-completeness for finite simple folds (no matter the number of layers) of unassigned orthogonal crease patterns on arbitrary paper, complementing a previous result for assigned crease patterns, and contrasting with a previous positive result for infinite all-layers simple folds. In total, we obtain a characterization of polynomial vs. NP-hard for all cases — finite/infinite one/some/all-layers simple folds of assigned/unassigned/mixed orthogonal crease patterns on 1D/rectangular/arbitrary paper — except the unsolved case of infinite all-layers simple folds of assigned orthogonal crease patterns on arbitrary paper.

Keywords: computational origami, simple folding, NP-hardness, algorithms.

2010 MSC: Primary 68Q25; Secondary 52C99.

1 Introduction

In the well-studied *simple foldability problem* [1–3], we are given a *crease pattern* consisting of line-segment *creases*, possibly *assigned* mountain or valley, on a 2D region called the *piece of paper*, and are asked whether all of the creases can be folded via a sequence of "simple folds". Each *simple fold* folds some set of layers of the piece of paper around a single line by $\pm 180°$ (thus preserving flatness of the folding). In the *1D* variation, the piece of paper is a 1D line segment and the creases are points.

Many different models for simple folds and special cases for the simple foldability problem have been considered, depending on the following characteristics:

- How many layers of paper a simple fold can move at once. The most powerful model, *some-layers*, allows the top or bottom k layers to be folded for any k. Two more restrictive models are *one-layer*, which permits folding only a single layer at a time (modeling thick material); and *all-layers*, which requires folding all layers simultaneously.

- Whether simple folds can be along *finite* line segments (chords of the crease pattern), or must be along *infinite* lines (modeling a half-plane/large flipping tool).

- Whether some creases are assigned as needing to be folded mountain or valley. In the more common *assigned* and *unassigned* cases, all creases either have or lack an assignment, while in the *mixed* case [1], each crease may be mountain, valley, or unassigned.

- What shape the paper is allowed to have. In 1D, the only option is a line segment, while in 2D the two cases considered are rectangles and arbitrary polygons, though all NP-hardness results hold even for simple orthogonal polygons.

		1D	$\subset$	Rectangular	$\subset$	Arbitrary Paper		
						Infinite	&	Finite
One Layer	Assigned	poly	$\Leftarrow$	poly [3]		NP-comp. [2]		NP-comp. [2]
	$\cap$	$\Uparrow$		$\Uparrow$		$\Downarrow$		$\Downarrow$
	Mixed	**poly** (R1)	$\Leftarrow$	**poly** (R1)		NP-comp.		NP-comp.
	$\cup$	$\Downarrow$		$\Downarrow$		$\Uparrow$		$\Uparrow$
	Unassigned	poly	$\Leftarrow$	poly [3]		NP-comp. [2]		**NP-comp.** (R4)
Some Layers	Assigned	poly	$\Leftarrow$	poly [3]		NP-comp. [2]		NP-comp. [2]
	$\cap$	$\Uparrow$				$\Downarrow$		$\Downarrow$
	Mixed	**poly** (R1)		**NP-comp.** (R3)	$\Rightarrow$	NP-comp.	&	NP-comp.
	$\cup$	$\Downarrow$				$\Uparrow$		$\Uparrow$
	Unassigned	poly	$\Leftarrow$	poly [3]		NP-comp. [2]		**NP-comp.** (R4)
All Layers	Assigned	poly	$\Leftarrow$	poly [3]		OPEN		NP-comp. [2]
	$\cap$	$\Uparrow$						$\Downarrow$
	Mixed	**poly** (R2)		NP-comp. [1]	$\Rightarrow$	NP-comp.	&	NP-comp.
	$\cup$	$\Downarrow$						$\Uparrow$
	Unassigned	poly	$\Leftarrow$	poly [3]	$\Leftarrow$	poly [1]		**NP-comp.** (R4)

Table 1: Summary of past results (cited) and new results (bold, with result numbers) about simple folding orthogonal crease patterns. "NP-comp." denotes strong NP-completeness. "$\subset$" denotes containment between classes of instances. "$\Rightarrow$" denotes implications between results. "R" numbers refer to the results list in Section 2. Hardness results for arbitrary paper hold even for paper a simple orthogonal polygon.

We consider exclusively *orthogonal crease patterns*, which contain only vertical and horizontal creases. Prior work [2, 3] has also studied versions of the problem including diagonal creases.

Table 1 presents all known results according to these parameters, including both previously known results and new results from this paper (in bold). Note that, for orthogonal crease patterns, the finite vs. infinite simple fold distinction only plays a role with arbitrary paper: no distinction can be made in the 1D case, and equivalence for rectangular paper is given by [2, Theorem 8].

2 Results

Our new results can be summarized as follows:

1. **One- & some-layers mixed 1D; one-layer mixed rectangular.** We adapt arguments from [3], and provide a new characterization of when a 1D mixed crease pattern is flat-foldable. For one-layer rectangular, crossing creases can never be folded, so the problem reduces to 1D paper.

2. **All-layers mixed 1D.** We show that folding the superficially foldable (foldable ignoring assignments) crease nearest an end of the paper never makes the crease pattern unfoldable, giving an efficient greedy strategy for deciding simple foldability.

3. **Some-layers mixed rectangular.** We show that the NP-hardness reduction from [1] for all-layers mixed rectangular also works for some-layers.

4. **Unassigned arbitrary finite.** We modify the NP-hardness proof from [2] for assigned crease patterns to work for unassigned crease patterns, by adding a component that enforces the relevant aspects of the assignment. The original proof and our adaptation apply to any number of layers.

Notably, the only case that remains unsolved for orthogonal crease patterns is infinite all-layers simple folds of assigned crease patterns on arbitrary paper.

References

[1] Hugo A. Akitaya, Cordelia Avery, Joseph Bergeron, Erik D. Demaine, Justin Kopinsky, and Jason S. Ku, *Infinite all-layers simple foldability*, Graphs and Combinatorics **36** (2020), 231–244, Special issue from JCDCGGG 2017.

[2] Hugo A. Akitaya, Erik D. Demaine, and Jason S. Ku, *Simple folding is really hard*, Journal of Information Processing **25** (2017), 580–589, Special issue from JCDCGGG 2016.

[3] Esther M. Arkin, Michael A. Bender, Erik D. Demaine, Martin L. Demaine, Joseph S. B. Mitchell, Saurabh Sethia, and Steven S. Skiena, *When can you fold a map?*, Computational Geometry **29** (2004), no. 1, 23–46.