Exponential Speedup of Fixed-Parameter Algorithms on $K_{3,3}$-minor-free or K_5-minor-free Graphs*

Erik D. Demaine[1], MohammadTaghi Hajiaghayi[1], and Dimitrios M. Thilikos[2]

[1] Laboratory for Computer Science, Massachusetts Institute of Technology, 200 Technology Square, Cambridge, MA 02139, U.S.A.

[2] Departament de Llenguatges i Sistemes Informàtics, Universitat Politècnica de Catalunya, Campus Nord – Mòdul C5, Desp. 211b, c/Jordi Girona Salgado, 1-3. E-08034, Barcelona, Spain

Abstract. We present a fixed-parameter algorithm that constructively solves the k-dominating set problem on graphs excluding one of K_5 or $K_{3,3}$ as a minor in time $O(4^{16.5\sqrt{k}}n^{O(1)})$, which is an exponential factor faster than the previous $O(2^{O(k)}n^{O(1)})$. In fact, we present our algorithm for any H-minor-free graph where H is a single-crossing graph (can be drawn in the plane with at most one crossing) and obtain the algorithm for $K_{3,3}(K_5)$-minor-free graphs as a special case. As a consequence, we extend our results to several other problems such as vertex cover, edge dominating set, independent set, clique-transversal set, kernels in digraphs, feedback vertex set and a series of vertex removal problems. Our work generalizes and extends the recent result of exponential speedup in designing fixed-parameter algorithms on planar graphs by Alber et al. to other (nonplanar) classes of graphs.

1 Introduction

According to a 1998 survey book [19], there are more than 200 published research papers on solving domination-like problems on graphs. Since this problem is very hard and NP-complete even for special kinds of graphs such as planar graphs, much attention has focused on solving this problem on a more restricted class of graphs. It is well known that this problem can be solved on trees [10] or even the generalization of trees, graphs of bounded treewidth [26]. The approximability of the dominating set problem has received considerable attention, but it is not known and it is not believed that this problem has constant factor approximation algorithms on general graphs [5].

* The work of the third author was supported by the IST Programme of the EU under contract number IST-1999-14186 (ALCOM-FT), the Spanish CICYT project TIC2000-1970-CE, and the Ministry of Education and Culture of Spain (Resolución 31/7/00 – BOE 16/8/00). Emails: {hajiagha,edemaine}@theory.lcs.mit.edu, and sedthilk@lsi.upc.es

Downey and Fellows [14] introduced a new concept to handle NP-hardness called *fixed-parameter tractability*. Unfortunately, according to this theory, it is very unlikely that the k-dominating set problem has an efficient fixed-parameter algorithm for general graphs. In contrast, this problem is fixed-parameter tractable on planar graphs. The first algorithm for planar k-dominating set was developed in the book of Downey and Fellows [14]. Recently, Alber et al. [1] demonstrated a solution to the planar k-dominating set in time $O(4^{6\sqrt{34k}}n)$ (for an improvement of this result, proposed by Kanj and Perković, see [20]). Indeed, this result was the first nontrivial result for the parameterized version of an NP-hard problem where the exponent of the exponential term grows sublinearly in the parameter. One of the aims of this paper is to generalize this result to nonplanar classes of graphs.

A graph G is *H-minor-free* if H cannot be obtained from any subgraph of G by contracting edges. A graph is called a *single-crossing graph* if it can be drawn in the plane with at most one crossing. Similar to the approach of Alber et al., we prove that for a single-crossing graph H, the treewidth of any H-minor-free graph G having a k-dominating set is bounded by $O(\sqrt{k})$. As a result, we generalize current exponential speedup in fixed-parameter algorithms on planar graphs to other kinds of graphs and show how we can solve the k-dominating set problem on $K_{3,3}$-minor-free or K_5-minor-free graphs in time $O(4^{16.5\sqrt{k}}n^{O(1)})$. The genesis of our results lies in a result of Hajiaghayi et al. [18] on obtaining the local treewidth of the aforementioned class of graphs. The classes of $K_{3,3}$-minor-free graphs and K_5-minor-free graphs have been considered before, e.g. in [22, 27].

Using the solution for the k-dominating set problem on planar graphs, Kloks et al. [9, 17, 23, 24] and Alber et al. [1, 2] obtained exponential speedup in solving other problems such as vertex cover, independent set, clique-transversal set, kernels in digraph and feedback vertex set on planar graphs. In this paper we also show how our results can be extended to these problems and many other problems such as variants of dominating set, edge dominating set and a series of vertex removal problems. The reader is referred to [12] for the full proofs of theorems in this paper.

2 Background

We assume the reader is familiar with general concepts of graph theory such as (un)directed graphs, trees and planar graphs. The reader is referred to standard references for appropriate background [8]. In addition, for exact definitions of various NP-hard graph-theoretic problems in this paper, the reader is referred to Garey and Johnson's book on computers and intractability [16].

Our graph terminology is as follows. All graphs are finite, simple and undirected, unless indicated otherwise. A graph G is represented by $G = (V, E)$, where V (or $V(G)$) is the set of vertices and E (or $E(G)$) is the set of edges. We denote an edge e in a graph G between u and v by $\{u, v\}$. We define n to be the number of vertices of a graph when it is clear from context. We define the *r-neighborhood* of a set $S \subseteq V(G)$, denoted by $N_G^r(S)$, to be the set of vertices at distance at most r from at least one vertex of $S \subseteq V(G)$; if $S = \{v\}$ we simply use the notation $N_G^r(v)$. The *union* of two disjoint graphs G_1 and G_2, $G_1 \cup G_2$, is a graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

For generalizations of algorithms on undirected graphs to directed graphs, we consider underlying graphs of directed graphs. The *underlying graph* of a directed graph H is the undirected graph G in which $V(G) = V(H)$ and $\{u, v\} \in E(G)$ if and only if $(u, v) \in E(H)$ or $(v, u) \in E(H)$.

Contracting an edge $e = \{u, v\}$ is the operation of replacing both u and v by a single vertex w whose neighbors are all vertices that were neighbors of u or v, except u and v themselves. A graph G is a *minor* of a graph H if H can be obtained from a subgraph of G by contracting edges. A graph class $\mathcal{C}$ is a *minor-closed* class if any minor of any graph in $\mathcal{C}$ is also a member of $\mathcal{C}$. A minor-closed graph class $\mathcal{C}$ is *H-minor-free* if $H \notin \mathcal{C}$. For example, a planar graph is a graph excluding both $K_{3,3}$ and K_5 as minors.

A *tree decomposition* of a graph $G = (V, E)$, denoted by $TD(G)$, is a pair (χ, T) in which $T = (I, F)$ is a tree and $\chi = \{\chi_i | i \in I\}$ is a family of subsets of $V(G)$ such that:

1. $\bigcup_{i \in I} \chi_i = V$;
2. $\forall_{e=\{u,v\} \in E}$ there exists an $i \in I$ such that both u and v belong to χ_i; and
3. $\forall_{v \in V}$, the set of nodes $\{i \in I | v \in \chi_i\}$ forms a connected subtree of T.

3 General results on clique-sum graphs

Suppose G_1 and G_2 are graphs with disjoint vertex-sets and $k \geq 0$ is an integer. For $i = 1, 2$, let $W_i \subseteq V(G_i)$ form a clique of size k and let G_i' ($i = 1, 2$) be obtained from G_i by deleting some (possibly no) edges from $G_i[W_i]$ with both endpoints in W_i. Consider a bijection $h : W_1 \rightarrow W_2$. We define a *k-sum* G of G_1 and G_2, denoted by $G = G_1 \oplus_k G_2$ or simply by $G = G_1 \oplus G_2$, to be the graph obtained from the union of G_1' and G_2' by identifying w with $h(w)$ for all $w \in W_1$. The images of the vertices of W_1 and W_2 in $G_1 \oplus_k G_2$ form the *join set*. In the rest of this section, when we refer to a vertex v of G in G_1 or G_2, we mean the corresponding vertex of v in G_1 or G_2 (or both). It is worth mentioning that $\oplus$ is not a well-defined operator and it can have a set of possible results.

Let s be an integer where $0 \leq s \leq 3$ and $\mathcal{C}$ be a finite set of graphs. We say that a graph class $\mathcal{G}$ is a *clique-sum class* if any of its graphs can be constructed by a sequence of i-sums ($i \leq s$) applied to planar graphs and graphs in $\mathcal{C}$. We call a graph *clique-sum* if it is a member of a clique-sum class. We call the pair $(\mathcal{C}, s)$ the *defining pair* of $\mathcal{G}$ and we call the maximum treewidth of graphs in $\mathcal{C}$ the *base* of $\mathcal{G}$ and the *base* of graphs in $\mathcal{G}$. A series of k-sums (not necessarily unique) which generate a clique-sum graph G are called *a decomposition of G into clique-sum operations*.

According to the result of [25], if $\mathcal{G}$ is the class of graphs excluding a single crossing graph (can be drawn in the plane with at most one crossing) H then $\mathcal{G}$ is a clique-sum class with defining pair $(\mathcal{C}, s)$ where the base of $\mathcal{G}$ is bounded by a constant c_H depending only on H. In particular, if $H = K_{3,3}$, the defining pair is $(\{K_5\}, 2)$ and $c_H = 4$ [28] and if $H = K_5$ then the defining pair is $(\{V_8\}, 3)$ and $c_H = 4$ [28]. Here by V_8 we mean the graph obtained from a cycle of length eight by joining each pair of diagonally opposite vertices by an edge. For more results on clique-sum classes see [13].

From the definition of clique-sum graphs, one can observe that, for any clique-sum graph G which excludes a single crossing graph H as a minor, any minor G' of G is also a clique-sum graph which excludes the same graph H as a minor.

We call a clique-sum graph class $\mathcal{G}$ *α-recognizable* if there exists an algorithm that for any graph $G \in \mathcal{G}$ outputs in $O(n^\alpha)$ time a sequence of clique sums of graphs of total size $O(|V(G)|)$ that constructs G. We call a graph *α-recognizable* if it belongs in some α-recognizable clique-sum graph class. Using the results in [21] and [4] one can verify the following.

Theorem 1 ([21, 4]). *The class of K_5-minor-free ($K_{3,3}$-minor-free) graphs is a 2-recognizable (1-recognizable) clique-sum class.*

A *parameterized graph class* (or just *graph parameter*) is a family $\mathcal{F}$ of classes $\{\mathcal{F}_i, i \geq 0\}$ where $\bigcup_{i \geq 0} \mathcal{F}_i$ is the set of all the graphs and for any $i \geq 0$, $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$. Given two parameterized graph classes $\mathcal{F}^1$ and $\mathcal{F}^2$ and a natural number $\gamma \geq 1$ we say that $\mathcal{F}^1 \preccurlyeq_\gamma \mathcal{F}^2$ if for any $i \geq 0$, $\mathcal{F}^1_i \subseteq \mathcal{F}^2_{\gamma \cdot i}$.

In the rest of this paper, we will identify a parameterized problem with the *parameterized graph class* corresponding to its "yes" instances.

Theorem 2. *Let $\mathcal{G}$ be an α-recognizable clique-sum graph class with base c and let $\mathcal{F}$ be a parameterized graph class. In addition, we assume that each graph in $\mathcal{G}$ can be constructed using i-sums where $i \leq s \leq 3$. Suppose also that there exist two positive real numbers β_1, β_2 such that:*

(1) For any $k \geq 0$, planar graphs in $\mathcal{F}_k$ have treewidth at most $\beta_1\sqrt{k} + \beta_2$ and such a tree decomposition can be found in linear time.
(2) For any $k \geq 0$ and any $i \leq s$, if $G_1 \oplus_i G_2 \in \mathcal{F}_k$ then $G_1, G_2 \in \mathcal{F}_k$

Then, for any $k \geq 0$, the graphs in $\mathcal{G} \cap \mathcal{F}_k$ all have treewidth $\leq \max\{\beta_1\sqrt{k} + \beta_2, c\}$ and such a tree decomposition can be constructed in $O(n^\alpha + (\sqrt{k})^s \cdot n)$ time.

Theorem 3. *Let $\mathcal{G}$ be a graph class and let $\mathcal{F}$ be some parameterized graph class. Suppose also for some positive real numbers $\alpha, \beta_1, \beta_2, \delta$ the following hold:*

(1) For any $k \geq 0$, the graphs in $\mathcal{G} \cap \mathcal{F}_k$ all have treewidth $\leq \beta_1\sqrt{k} + \beta_2$ and such a tree decomposition can be decided and constructed (if it exists) in $O(n^\alpha)$ time. We also assume testing membership in $\mathcal{G}$ takes $O(n^\alpha)$ time.
(2) Given a tree decomposition of width at most w of a graph, there exists an algorithm deciding whether the graph belongs in $\mathcal{F}_k$ in $O(\delta^w n)$ time.

Then there exists an algorithm deciding in $O(\delta^{\beta_1\sqrt{k}+\beta_2} n + n^\alpha)$ time whether an input graph G belongs in $\mathcal{G} \cap \mathcal{F}_k$.

It is worth mentioning that Demaine et al. [11] very recently designed a polynomial-time algorithm to decompose any H-minor-free graph, where H is a single-crossing graph, into clique-sum operations. Thus $O(n^{\alpha})$ is polynomial for these H-minor-free graphs.

4 Fixed-parameter algorithms for dominating set

In this section, we will describe some of the consequences of Theorems 2 and 3 on the design of efficient fixed-parameter algorithms for a series of parameterized problems where their inputs are clique-sum graphs.

A *dominating set* of a graph G is a set of vertices of G such that each of the rest of vertices has at least one neighbor in the set. We represent the k-*dominating set* problem with the parameterized graph class $\mathcal{DS}$ where $\mathcal{DS}_k$ contains graphs which have a dominating set of size $\leq k$. Our target is to show how we can solve the k-dominating set problem on clique-sum graphs, where H is a single-crossing graph, in time $O(c^{\sqrt{k}} n^{O(1)})$ instead of the current algorithms which run in time $O(c^k n^{O(1)})$ for some constant c. By this result, we extend the current exponential speedup in designing algorithms for planar graphs [2] to a more generalized class of graphs. In fact, planar graphs are both $K_{3,3}$-minor-free and K_5-minor-free graphs, where both $K_{3,3}$ and K_5 are single-crossing graphs.

According to the results in [20], condition (1) of Theorem 2 is satisfied for $\mathcal{DS}$ for $\beta_1 = 16.5$ and $\beta_2 = 50$. Also, the next lemma shows that condition (2) of Theorem 2 is also correct.

Lemma 1. *If $G = G_1 \oplus_m G_2$ has a k-dominating set, then both G_1 and G_2 have dominating sets of size at most k.*

We can now apply Theorem 2 for $\beta_1 = 16.5$ and $\beta_2 = \max\{50, c\}$.

Theorem 4. *If $\mathcal{G}$ is an α-recognizable clique-sum class of base c, then any member G of $\mathcal{G}$ where its dominating set has size at most k has treewidth at most $16.5\sqrt{k} + \max\{50, c\}$ and the corresponding tree decomposition of G can be constructed in $O(n^{\alpha})$ time.*

Theorem 4 tells us that condition (1) of Theorem 3 is satisfied. Moreover, according to the results in [1, 3] condition (2) of Theorem 3, is satisfied for the graph parameter $\mathcal{DS}$ when $\delta = 4$. Applying now theorems 3 and 4 we have the following.

Theorem 5. *There is an algorithm that in $O(4^{16.5\sqrt{k}} n + n^{\alpha})$ time solves the k-dominating set problem for any α-recognizable clique-sum graph. Consequently, there is an algorithm that in $O(4^{16.5\sqrt{k}} n)$ $(O(4^{16.5\sqrt{k}} n + n^2))$ time solves the k-dominating set problem for $K_{3,3}(K_5)$-minor-free graphs.*

5 Algorithms for parameters bounded by dominating set number

We provide a general methodology for deriving fast fixed-parameter algorithms in this section. First, we consider the following theorem which is an immediate consequence of Theorem 3.

Theorem 6. *Let $\mathcal{G}$ be a graph class and let $\mathcal{F}^1$, $\mathcal{F}^2$ be two parameterized graph classes where $\mathcal{F}^1 \preccurlyeq_\gamma \mathcal{F}^2$ for some natural number $\gamma \geq 1$. Suppose also that there exist positive real numbers $\alpha, \beta_1, \beta_2, \delta$ such that:*

(1) For any $k \geq 0$, the graphs in $\mathcal{G} \cap \mathcal{F}^2_k$ all have treewidth $\leq \beta_1\sqrt{k} + \beta_2$ and such a tree decomposition can be decided and constructed (if it exists) in $O(n^\alpha)$ time. We also assume testing membership in $\mathcal{G}$ takes $O(n^\alpha)$ time.
(2) There exists an algorithm deciding whether a graph of treewidth $\leq w$ belongs in $\mathcal{F}^1_k$ in $O(\delta^w n)$ time.
Then:
(1) For any $k \geq 0$, the graphs in $\mathcal{G} \cap \mathcal{F}^1_k$ all have treewidth at most $\beta_1\sqrt{\gamma k} + \beta_2$ and such a tree decomposition can be constructed in $O(n^\alpha)$ time.
(2) There exists an algorithm deciding in $O(\delta^{\beta_1\sqrt{\gamma k}+\beta_2} n + n^\alpha)$ time whether an input graph G belongs in $\mathcal{G} \cap \mathcal{F}^1_k$.

The idea of our general technique is given by the following theorem that is a direct consequence of Theorems 4 and 6.

Theorem 7. *Let $\mathcal{F}$ be a parameterized graph class satisfying the following two properties:*

(1) It is possible to check membership in $\mathcal{F}_k$ of a graph G of treewidth at most w in $O(\delta^w n)$ time for some positive real number δ.
(2) $\mathcal{F} \preccurlyeq_\gamma \mathcal{DS}$.
Then:
(1) Any clique sum graph G of base c in $\mathcal{F}_k$ has treewidth at most $\max\{16.5\sqrt{\gamma k} + 50, c\}$.
(2) We can check whether an input graph G is in $\mathcal{F}_k$ in $O(\delta^{16.5\sqrt{\gamma k}} n + n^\alpha)$ [1] on an α-recognizable clique-sum graph of base c.

Theorem 7 applies for several graph parameters. In particular it can be applied for the k-weighted dominated set problem, the k-dominating set problem with property Π, the Y-domination problem, the k-vertex cover problem, the k-edge dominating set problem, the k-edge trasversal set problem, the minimum maximum matching problem, the k-kernel problem in digraphs and the k-independent set problem. For more details on the analysis of each of these problems, see [12].

[1] In the rest of this paper, we assume that constants, e.g. c, are small and they do not appear in the powers, since they are absorbed into the O notation.

6 Fixed-parameter algorithms for vertex removal problems

For any graph class $\mathcal{G}$ and any nonnegative integer k the graph class k-almost*($\mathcal{G}$)* contains any graph $G = (V, E)$ where there exists a subset $S \subseteq V(G)$ of size at most k such that $G[V - S] \in \mathcal{G}$. We note that using this notation if $\mathcal{G}$ contains all the edgeless graphs or forests then k-almost($\mathcal{G}$) is the class of graphs with vertex cover $\leq k$ or feedback vertex set $\leq k$.

A graph $G = (V, E)$ has a *k-cut* $S \subseteq V$ when $G[V - S]$ is disconnected and $|S| = k$. Let G_1, G_2 be two of the connected components of $G[V - S]$. Given a component $G_1 = (V_1, E_1)$ of $G[V - S]$ we define its *augmentation* as the graph $G[V_1 \cup S]$ in which we add all edges among vertices of S. We say a k-cut S *minimally separates* G_1 and G_2 if each vertex of S has a neighbor in G_1 and G_2. A graph $G = (V, E)$ has a *strong k-cut* $S \subseteq V$ if $|S| = k$ and $G[V - S]$ has at least k connected components and each pair of them is minimally separated by S. We say that G is the result of the multiple k-clique sum of $G_1, \ldots, G_r$ with respect to some join set W if $G = G_1 \oplus_k \cdots \oplus_k G_r$ where the join set is always W and such that W is a strong k-cut of G.

Lemma 2. *Let k be a positive integer and let G be a graph with a strong k-cut S where $1 \leq k$. Then the treewidth of G is bounded above by the maximum of the treewidth of each of the augmented components of G after removing S.*

Lemma 3. *Let $G = (V, E)$ be a graph with a strong k-cut S where $1 \leq k \leq 3$. Then if G belongs to some minor-closed graph class $\mathcal{G}$ then any of the augmented components of G after removing S is also k-connected and belongs to $\mathcal{G}$.*

We now need the following adaptation of the results of [21] and [4] (Theorem 1).

Lemma 4. *Let G be a connected $K_{3,3}$-free graph and let $\mathcal{S}$ be the set of its strong i-cuts, $1 \leq i \leq 2$. Then G can be constructed after a sequence of multiple i-clique sums, $1 \leq i \leq 2$, applied to planar graphs or K_5's where each of these multiple sums has a member of $\mathcal{S}$ as join set. Moreover this sequence can be constructed by an algorithm in $O(n)$ time.*

Lemma 5. *Let G be a connected K_5-free graph and let $\mathcal{S}$ be the set of its strong i-cuts, $1 \leq i \leq 3$. Then G can be constructed after a sequence of multiple i-clique sums, $1 \leq i \leq 3$, applied to planar graphs or V_8's where each of these multiple sums has a member of $\mathcal{S}$ as join set. Moreover this sequence can be constructed by an algorithm in $O(n^2)$ time.*

Theorem 8. *Let $\mathcal{G}$ be a $K_{3,3}$(K_5)-minor-free graph class and let $\mathcal{F}$ be any minor-closed parameterized graph class. Suppose that there exist real numbers $\beta_0 \geq 4, \beta_1$ such that any planar graph in $\mathcal{F}_k$ has treewidth at most $\beta_1\sqrt{k} + \beta_0$ and such a tree decomposition can be found in linear time. Then graphs in $\mathcal{G} \cap \mathcal{F}_k$ all have treewidth $\leq \beta_1\sqrt{k} + \beta_0$ and such a tree decomposition can be constructed in $O(n)$ ($O(n^2)$) time.*

We define $\mathcal{T}_r$ to be the class of graphs with treewidth $\leq r$. It is known that for $1 \leq i \leq 2$, $\mathcal{T}_i$ is exactly the class of K_{i+2}-minor-free graphs.

Lemma 6. *Planar graphs in* k-almost$(\mathcal{T}_2)$ *have treewidth* $\leq 16.5\sqrt{k}+50$*. Moreover, such a tree decomposition can be found in linear time.*

We conclude the following general result:

Theorem 9. *Let* $\mathcal{G}$ *be any class of graphs with treewidth* ≤ 2*. Then any* $K_{3,3}(K_5)$*-minor-free graph in* k-almost$(\mathcal{G})$ *has treewidth* $\leq 16.5\sqrt{k}+50$*. Moreover, such a tree decomposition can be found in* $O(n)$ $(O(n^2))$ *time.*

Combining Theorems 3 and 9 we conclude the following.

Theorem 10. *Let* $\mathcal{G}$ *be any class of graphs with treewidth* ≤ 2*. Suppose also that there exists an* $O(\delta^w n)$ *algorithm that decides whether a given graph belongs in* k-almost$(\mathcal{G})$ *for graphs of treewidth at most* w*. Then, one can decide whether a* $K_{3,3}(K_5)$*-minor-free graph belongs in* k-almost$(\mathcal{G})$ *in time* $O(\delta^{16.5\sqrt{k}}n + n^{\alpha})$*.*

If $\{O_1, \ldots, O_r\}$ is a finite set of graphs, we denote as minor-excl$(O_1, \ldots, O_r)$ the class of graphs that are O_i-minor-free for $i = 1, \ldots, r$.

As examples of problems for which Theorems 9 and 10 can be applied, we mention the problems of checking whether a graph, after removing k vertices, is *edgeless* ($\mathcal{G} = \mathcal{T}_0$), or has *maximum degree* ≤ 2 ($\mathcal{G} =$ minor-excl$(K_{1,3})$), or becomes a *a star forest* ($\mathcal{G} =$ minor-excl(K_3, P_3)), or a *caterpillar* ($\mathcal{G} =$ minor-excl(K_3, subdivision of $K_{1,3}$)), or a *forest* ($\mathcal{G} = \mathcal{T}_1$), or *outerplanar* ($\mathcal{G} =$ minor-excl$(K_4, K_{2,3})$), or *series-parallel*, or has treewidth $\leq k$ ($\mathcal{G} = \mathcal{T}_2$).

We consider the cases where $\mathcal{G} = \mathcal{T}_0$ and $\mathcal{G} = \mathcal{T}_1$. In particular we prove the following (for details, see [12]).

Theorem 11. *For any* $K_{3,3}(K_5)$*-minor-free graph* G *the following hold.*
(1) If G *has a feedback vertex set of size at most* k *then* G *has treewidth at most* $16.5\sqrt{k}+50$*.*
(2) We can check whether G *has a feedback vertex set of size* $\leq k$ *in* $O(c_{\mathsf{fvs}}^{16.5\sqrt{k}}n + n)$ $(O(c_{\mathsf{fvs}}^{16.5\sqrt{k}}n + n^2))$ *time, for some small constant* c_{fvs}*.*
(3) If G *has a vertex cover of size at most* k *then* G *has treewidth at most* $4\sqrt{3}\sqrt{k}+5$*.*
(4) We can check whether G *has a vertex cover of size* $\leq k$ *in* $O(2^{4\sqrt{3}\sqrt{k}}n + n)$ $(O(2^{4\sqrt{3}\sqrt{k}}n + n^2))$ *time.*
(5) We can check whether G *has a vertex cover of size* $\leq k$ *in* $O(2^{4\sqrt{3}\sqrt{k}}k + kn + n)$ $(O(2^{4\sqrt{3}\sqrt{k}}k + kn + n^2))$ *time.*

7 Further extensions

In this section, we obtain fixed-parameter algorithms with exponential speedup for k-vertex cover and k-edge dominating set on graphs more general than

$K_{3,3}(K_5)$-minor-free graphs. Our approach, similar to the Alber et al.'s approach [2], is a general one that can be applied to other problems.

Baker [6] developed several approximation algorithms to solve NP-complete problems for planar graphs. To extend these algorithms to other graph families, Eppstein [15] introduced the notion of bounded local treewidth, defined formally below, which is a generalization of the notion of treewidth. Intuitively, a graph has bounded local treewidth (or locally bounded treewidth) if the treewidth of an r-neighborhood of each vertex $v \in V(G)$ is a function of r, $r \in \mathbb{N}$, and not $|V(G)|$.

The *local treewidth* of a graph G is the function $\text{ltw}^G : \mathbb{N} \to \mathbb{N}$ that associates with every $r \in \mathbb{N}$ the maximum treewidth of an r-neighborhood in G. We set $\text{ltw}^G(r) = \max_{v \in V(G)}\{\text{tw}(G[N_G^r(v)])\}$, and we say that a graph class $\mathcal{C}$ has *bounded local treewidth (or locally bounded treewidth)* when there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $G \in \mathcal{C}$ and $r \in \mathbb{N}$, $\text{ltw}^G(r) \leq f(r)$.

A graph is called an *apex graph* if deleting one vertex produces a planar graph. Eppstein [15] showed that a minor-closed graph class $\mathcal{E}$ has bounded local treewidth if and only if $\mathcal{E}$ is H-minor-free for some apex graph H.

So far, the only graph classes studied with small local treewidth are the class of planar graphs [15] and the class of clique-sum graphs, which includes minor-free graphs like $K_{3,3}$-minor-free or K_5-minor-free graphs [18]. It has been proved that for any planar graph G, $\text{ltw}^G(k) \leq 3k-1$ [18], and for any $K_{3,3}$-minor-free or K_5-minor-free graph G, $\text{ltw}^G(k) \leq 3k+4$ [15]. For these classes of graphs, there are efficient algorithms for constructing tree decompositions.

Eppstein [15] showed how the concept of the kth outer face in planar graphs can be replaced by the concept of the kth layer (or level) in graphs of locally bounded treewidth. The kth layer (L_k) of a graph G consists of all vertices at distance k from an arbitrary fixed vertex v of $V(G)$. We denote *consecutive layers from i to j* by $L[i,j] = \cup_{i \leq k \leq j} L_k$.

Here we generalize the concept of *layerwise separation*, introduced in Alber et al.'s work [2] for planar graphs, to general graphs.

Let G be a graph layered from a vertex v, and r be the number of layers. A *layerwise separation of width w and size s* for G is a sequence $(S_1, S_2, \cdots, S_r)$ of subsets of V, with property that $S_i \subseteq \bigcup_{j=i}^{i+(w-1)} L_j$; S_i separates layers L_{i-1} and L_{i+w}; and $\sum_{j=1}^{r} |S_j| \leq s$.

A parameterized problem P has *Layerwise Separation Property (LSP)* of width w and size-factor d, if for each instance (G,k) of the problem P, graph G admits a layerwise separation of width w and size dk.

For example, we can obtain constants $w = 2$ and $d = 2$ for the vertex cover problem. In fact, consider a k-vertex cover C on a graph G and set $S_i = (L_i \cup L_{i+1}) \cap C$. The S_i's form a layerwise separation. Similarly, we can get constants $w = 2$ and $d = 2$ for the edge dominating set problem.

Lemma 7. *Let P be a parameterized problem on instance (G,k) that admits a problem kernel of size dk. Then the parameterized problem P on the problem kernel has LSP of width 1 and size-factor d.*

In fact, using Lemma 7 and the problem kernel of size $2k$ for the vertex cover problem, this problem has the LSP of width 1 and size-factor 2.

The proof of the following theorem is very similar to the proof of Theorem 14 of Alber et al.'s work [2] and hence omitted.

Theorem 12. *Suppose for a graph G, $\mathrm{ltw}^G(r) \leq cr+d$ and a tree decomposition of width $ch+d$ can be constructed in $O(n^\alpha)$ for any h consecutive layers (h is a constant). Also assume G admits a layerwise separation of width w and size dk. Then we have $tw(G) \leq 2\sqrt{6dk} + cw + d$. Such a tree decomposition can be computed in time $O(n^\alpha)$.*

Now, since for any H-minor-free graph G, where H is a single-crossing graph, $\mathrm{ltw}^G(r) \leq 3r + c_H$ and $tw(L[i,j]) \leq 3(j-i+1)+c_H$ [18], we have the following.

Corollary 1. *For any H-minor-free graph G, where H is a single-crossing graph, that admits a layerwise separation of width w and size dk, we have $tw(G) \leq 2\sqrt{6dk} + 3w + c_H$.*

Since we can construct the aforementioned kind of tree decompositions for $K_{3,3}(K_5)$-minor-free graphs in $O(n)(O(n^2))$ and their local treewidth is $3r+4$ [18], the following result follows immediately.

Corollary 2. *For any $K_{3,3}(K_5)$-minor-free graph G, that admits a layerwise separation of width w and size dk, we have $tw(G) \leq 2\sqrt{6dk} + 3w + 4$. Such a tree decomposition can be computed in time $O(n)$ $(O(n^2))$.*

In fact, we have this general theorem.

Theorem 13. *Suppose for a graph G, $\mathrm{ltw}^G(r) \leq cr+d$ and a tree decomposition of width $ch+d$ can be constructed in time $O(n^\alpha)$ for any h consecutive layers. Let P be a parameterized problem on G such that P has the LSP of width w and size-factor d and there exists an $O(\delta^w n)$-time algorithm, given a tree decomposition of width w for G, decides whether problem P has a solution of size k on G.*

Then there exists an algorithm which decides whether P has a solution of size k on G in time $O(\delta^{2\sqrt{6dk}+cw+d} n + n^\alpha)$.

8 Conclusions and future work

In this paper, we considered H-minor-free graphs, where H is a single-crossing graph, and proved that if these graphs have a k-dominating set then their treewidth is at most $c\sqrt{k}$ for a small constant c. As a consequence, we obtained exponential speedup in designing FPT algorithms for several NP-hard problems on these graphs, especially $K_{3,3}$-minor-free or K_5-minor-free graphs. In fact, our approach is a general one that can be applied to several problems which can be reduced to the dominating set problem as discussed in Section 5 or to problems that themselves can be solved exponentially faster on planar graphs [2]. Here, we present several open problems that are possible extensions of this paper.

One topic of interest is finding other problems to which the technique of this paper can be applied. Moreover, it would be interested to find other classes of graphs than H-minor-free graphs, where H is a single-crossing graph, on which the problems can be solved exponentially faster for parameter k.

For several problems in this paper, Kloks et al. [9, 24, 17, 23] introduced a reduction to the problem kernel on planar graphs. Since $K_{3,3}$-minor-free graphs and K_5-minor-free graphs are very similar to planar graphs in the sense of having a linear number of edges and not having a clique of size six, we believe that one might obtain similar results for these graphs. Working in this area was beyond the scope of this paper, but still it would be instructive.

As mentioned before, Theorem 9 holds for any class of graphs with treewidth ≤ 2. It is an open problem whether it is possible to generalize it to apply to any class of graphs of treewidth $\leq h$ for arbitrary fixed h. Moreover, there exists no general method for designing $O(\delta^w n)$-time algorithms for vertex removal problems in graphs with treewidth $\leq w$. If this becomes possible, then Theorem 10 will have considerable algorithmic applications.

Finally, as a matter of practical importance, it would be interesting to obtain a constant coefficient better than 16.5 for the treewidth of planar graphs having a k-dominating set (or better than $4\sqrt{3}$ for the case of a k-vertex cover). Such a result would imply a direct improvement to our results and to all the results in [1, 2, 23, 9, 24].

Acknowledgments

We thank Prabhakar Ragde and Naomi Nishimura for their encouragement and help on this paper.

References

1. Jochen Alber, Hans L. Bodlaender, Henning Fernau, Ton Kloks, and Rolf Niedermeier. Fixed parameter algorithms for planar dominating set and related problems. *Algorithmica*, 33(4):461–493, 2002.
2. Jochen Alber, Henning Fernau, and Rolf Niedermeier. Parameterized complexity: Exponential speed-up for planar graph problems. In *Electronic Colloquium on Computational Complexity (ECCC)*. Germany, 2001.
3. Jochen Alber and Rolf Niedermeier. Improved tree decomposition algorithms for domination like problems. In *Latin American Theoretical Informatics*, pages 613–627. Lecture Notes in Computer Science, volume 2286, 2002.
4. Takao Asano. An approach to the subgraph homeomorphism problem. *Theoret. Comput. Sci.*, 38(2-3):249–267, 1985.
5. G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, and M. Protasi. *Complexity and approximation.* Springer-Verlag, Berlin, 1999.
6. Brenda S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *J. Assoc. Comput. Mach.*, 41(1):153–180, 1994.
7. Hans L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoret. Comput. Sci.*, 209(1-2):1–45, 1998.

8. John A. Bondy and U. S. R. Murty. *Graph Theory with Applications*. American Elsevier Publishing Co., Inc., New York, 1976.
9. Maw-Shang Chang, Ton Kloks, and Chuan-Min Lee. Maximum clique transversals. In *Proceedings of the 27th International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 300–310. Boltenhagen, Germany, 2001.
10. Ernest J. Cockayne, Sue Goodman, and Stephen Hedetniemi. A linear time algorithm for the domination number of a tree. *Inform. Process. Lett.*, 4(1):41–44, 1975.
11. Erik D. Demaine, Mohammadtaghi Hajiaghayi, and Dimitrios M. Thilikos. 1.5-Approximation for Treewidth of Graphs Excluding a Graph with One Crossing as a Minor. In *Proceedings of the 5th International Workshop on Approximation Algorithms for Combinatorial Optimization*, 2002, to appear.
12. Erik D. Demaine, Mohammadtaghi Hajiaghayi, and Dimitrios M. Thilikos. Exponential speedup of fixed parameter algorithms on $K_{3,3}$-minor-free or K_5-minor-free graphs. Technical Report MIT-LCS-TR-838, Mass. Inst. Technology, March 2002.
13. Reinhard Diestel. Simplicial decompositions of graphs: a survey of applications. *Discrete Math.*, 75(1-3):121–144, 1989.
14. Rodney G. Downey and Michael R. Fellows. *Parameterized Complexity*. Springer-Verlag, New York, 1999.
15. David Eppstein. Diameter and treewidth in minor-closed graph families. *Algorithmica*, 27(3-4):275–291, 2000.
16. Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-completeness*. W. H. Freeman and Co., 1979.
17. Gregory Gutin, Ton Kloks, and C. M. Lee. Kernels in planar digraphs. In *Optimization Online*. Mathematical Programming Society, Philadelphia, 2001.
18. Mohammadtaghi Hajiaghayi, Naomi Nishimura, Prabhakar Ragde, and Dimitrios M. Thilikos. Fast approximation schemes for $K_{3,3}$-minor-free or K_5-minor-free graphs. In *Euroconf. on Combinatorics, Graph Theory and Applications*, 2001.
19. Teresa W. Haynes, Stephen T. Hedetniemi, and Peter J. Slater. *Fundamentals of domination in graphs*. Marcel Dekker Inc., New York, 1998.
20. Iyad A. Kanj and Ljubomir Perković. Improved parameterized algorithms for planar dominating set. In *27th International Symposium on Mathematical Foundations of Computer Science*. Lecture Notes in Computer Science 2002.
21. André Kézdy and Patrick McGuinness. Sequential and parallel algorithms to find a K_5 minor. In *Proceedings of the Third Annual ACM-SIAM Symposium on Discrete Algorithms (Orlando, FL, 1992)*, pages 345–356, 1992.
22. Samir Khuller. Extending planar graph algorithms to $K_{3,3}$-free graphs. *Inform. and Comput.*, 84(1):13–25, 1990.
23. Ton Kloks and Leizhen Cai. Parameterized tractability of some (efficient) Y-domination variants for planar graphs and t-degenerate graphs. In *International Computer Symposium (ICS)*. Taiwan, 2000.
24. Ton Kloks, C.M. Lee, and Jim Liu. Feedback vertex sets and disjoint cycles in planar (di)graphs. In *Optimization Online*. Math. Prog. Soc., Philadelphia, 2001.
25. Neil Robertson and Paul Seymour. Excluding a graph with one crossing. In *Graph structure theory (Seattle)*, pages 669–675. Amer. Math. Soc., 1993.
26. Jan A. Telle and Andrzej Proskurowski. Practical algorithms on partial k-trees with an application to domination-like problems. In *Proceedings of Third Workshop on Algorithms and Data Structures (Montréal, 1993)*, pages 610–621.
27. Vijay V. Vazirani. NC algorithms for computing the number of perfect matchings in $K_{3,3}$-free graphs and related problems. *Inform. and Comput.*, 80(2):152–164, 1989.
28. Kehrer Wagner. Über eine Eigenschaft der eben Komplexe. *Deutsche Math.*, 2:280–285, 1937.