Design of Circular-Arc Curved Creases of Constant Fold Angle

Erik D. Demaine, Martin L. Demaine, and Klara Mundilova

CSAIL, MIT, Massachusetts, USA; {edemaine,mdemaine,kmundil}@mit.edu

Abstract

We present a tool for the design of curved folds of constant angle along curves composed of circular arcs. This tool allows the user to specify the intended shape, and computes its development. In general, the crease curves fold into helices of constant descent angle. If the crease curves are designed to remain planar (zero descent), our method additionally supports multiple pleated creases.

Introduction

The mathematics of folding sheets along curved creases has been developed over the past two decades [1, 2, 4]. The bulk of this work has focused on analyzing foldable crease patterns, in particular characterizing foldability, including locally around a crease and the regions between creases. Compared to straight-crease origami, there is relatively little work on algorithmic design of curved-crease foldings. In particular, existing curved-crease design software [5, 6] has focused on "forward" design, where the user manipulates the paper according to certain operations and the software models the result.

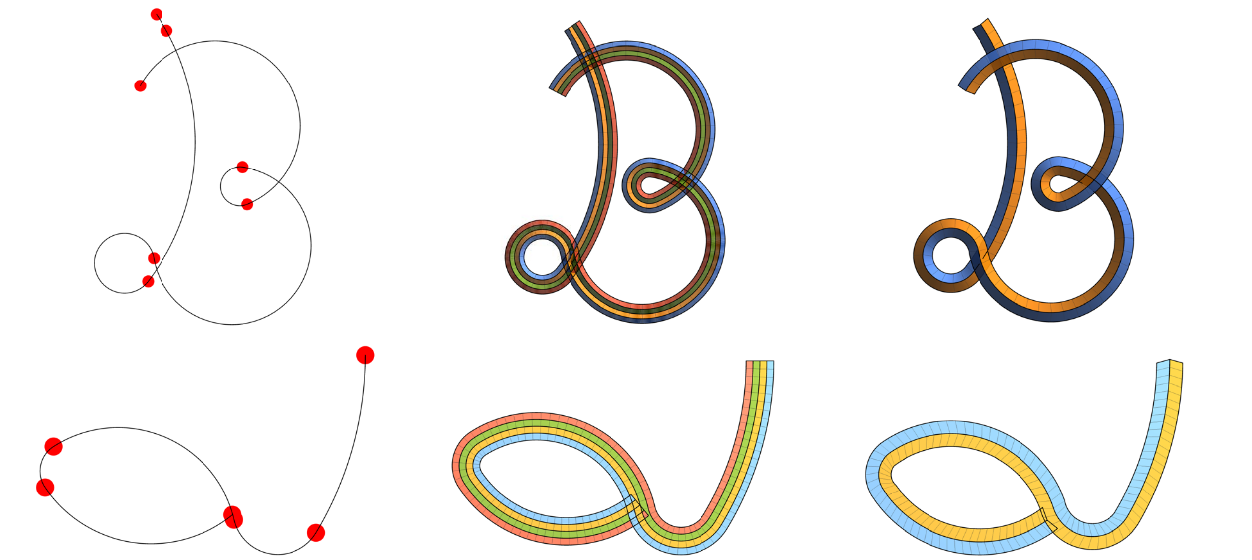
In this paper, we focus on "inverse" design, where the user specifies the paper's desired final form, and the software reverse-engineers the design. Unlike [7], we do so for a very special class of curved-crease designs, whose crease patterns consist of one or more pleated curved creases formed by circular arcs with matching tangents, and whose foldings have constant fold angle along the curve(s). Within this restriction, the mathematics is relatively clean, making the inverse problem relatively straightforward, thereby enabling the user to easily produce designs that fold into desired shapes. In addition to the case where the folding keeps the creases in a single plane, we analyze the more general setting where the creases form helices with a constant descent angle. Figure 1 shows shapes obtained by our method. The next section gives an intuitive construction, while the following section gives a formal mathematical description.

Folding strips of paper with one or more pleated curved creases has been explored before in the artistic setting. Notably, Josef Albers (of the Bauhaus) folded an elegant three-crease design while at Black Mountain College in the 1940s [8]. Jun Mitani features several one-crease designs in his recent book [9, pp. 6–11], which he calls "squiggles"; most impressive is his treble clef.

We developed an online web tool [3] implementing the basic computation of our method. The user can click a sequence of points defining a curve composed of circular arcs and lines for the folded state, and the software computes the unfolded development — the 2D crease that would fold into the specified planar curve. (Currently, the fold angle is assumed to be $\pi/3$.) We use our tool to develop a curved-crease font in the last section.

Intuitive Description of Folding with Helical Crease Curves

When deforming a sheet of paper without stretching, tearing or creasing, the resulting surface is composed of a family of straight lines. These lines are in particular visible as contours of the paper from different viewpoints. In the mathematical world, the surfaces that reflect this behaviour are compositions of so-called *developable surfaces*, which are surfaces that can be unrolled into the plane without stretching or tearing.



(a) *Input curve composed of circular arcs through given points and its possible canonical development.* **(b)** *Design with pleated planar circular creases with constant fold angle and its development.* **(c)** *Design with non-planar circular curved creases with constant angle and its development.*

Figure 1: *Illustration of a design obtained by the proposed method.*

Equivalently, they can be obtained by sweeping a line through space such that the tangent plane of the resulting surface is constant along this line. This reflects the property that these generating lines, also called *generators* or *rule lines*, are (local) contour lines of the surface. There are three types of non-planar developable surfaces: cylinders, cones and tangent surfaces, i.e., envelopes of tangents to a space curve. The basis of our design tool is to form circular and straight folds of constant angle.

Circles, helices and straight lines

Helices are curves that result from the combination of a translation and a rotation perpendicular to the direction of translation. More precisely, a helix can be parametrized by

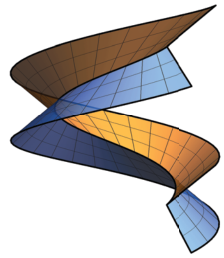
$$\mathbf{X}(t) = R(\cos t, \sin t, ht). \tag{1}$$

Note, that the points on $\mathbf{X}(t)$ lie on a right circular cylinder with radius R whose base is obtained for $h = 0$. If this cylinder, including the curve, is developed into the plane, the development $\mathbf{x}(t)$ of $\mathbf{X}(t)$ is a straight line with slope h. Furthermore, the tangents and the osculating planes of $\mathbf{X}(t)$ enclose a constant angle with respect to the z-axis.

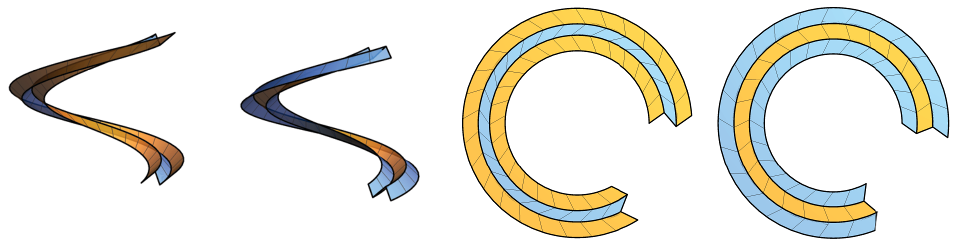
Gluing circular strips to helices

Suppose we are given a circular strip of paper bounded by concentric circles $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ with radii r_1 and r_2 with $R < r_1 < r_2$, then there are two options to glue this strip to the helix $\mathbf{X}(t)$ along the curve $\mathbf{x}_1$, both resulting in a developable surface. See Figure 2a. Due to the rotational nature of the curve, the developable surface's rulings will again enclose a constant angle with the base plane. There are two cases:

- If $\mathbf{X}(t)$ is a circle, these rulings will meet in a point, the tip of a right circular cone. The space curve corresponding to $\mathbf{x}_2(t)$ remains a circle.
- If $\mathbf{X}(t)$ is not a circle, these rulings compose a tangent surface whose edge of regression is a helix with same slope but smaller radius. The space curve corresponding to $\mathbf{x}_2(t)$ becomes another helix of same



(a) *Two circular strips of paper glued to a helix.*



(b) *Construction of pleated folds along helices by reassembling equidistantly cut surfaces and their developments.*

Figure 2: *Construction of pleated helical folds of constant angle.*

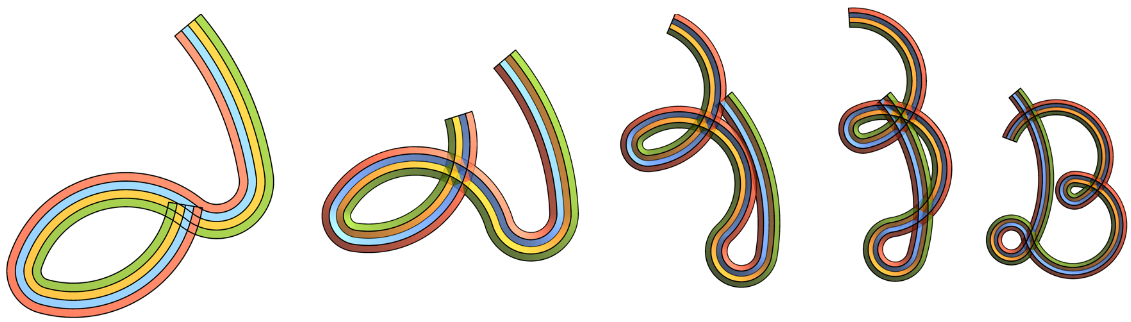


Figure 3: *Folding motion with rulings perpendicular to the crease curve.*

slope but greater radius.

Pleated circular and helical folds

From the two options of positions of a circular strip to a helix or a cone, we can construct pleated folds by cutting the surface at helices that share the same distance in the upper and lower surface. As depicted in Figure 2b, the hereby obtained thinner strips can be rearranged to construct pleated folds of constant fold angle. Note that in the case of circles these pleated cones are reflection on planes. Furthermore, this method does not impose any restrictions on the number of folds or the distance between the creases.

Combinations of circular strips glued to helices

Suppose that we arranged two circular strips along two helical curves. We show in the following section that we can join these strips if and only if the slope of the curves is the same and the ratios between the curvature of the spatial curve and developed curvature are equal. In addition, we can also combine the hereby constructed creases with straight segments whose direction corresponds to the tangent direction of the endpoints of the crease. See Figure 4 for an illustration.

Combinations of pleated folds

The above proposed construction of pleated folds of constant angle cannot be generalized to combinations of proper helices because of the transition area between the two segments. However, if the ruling directions are perpendicular as in the case of planar folds, we can construct pleated folds by reflections on planes.

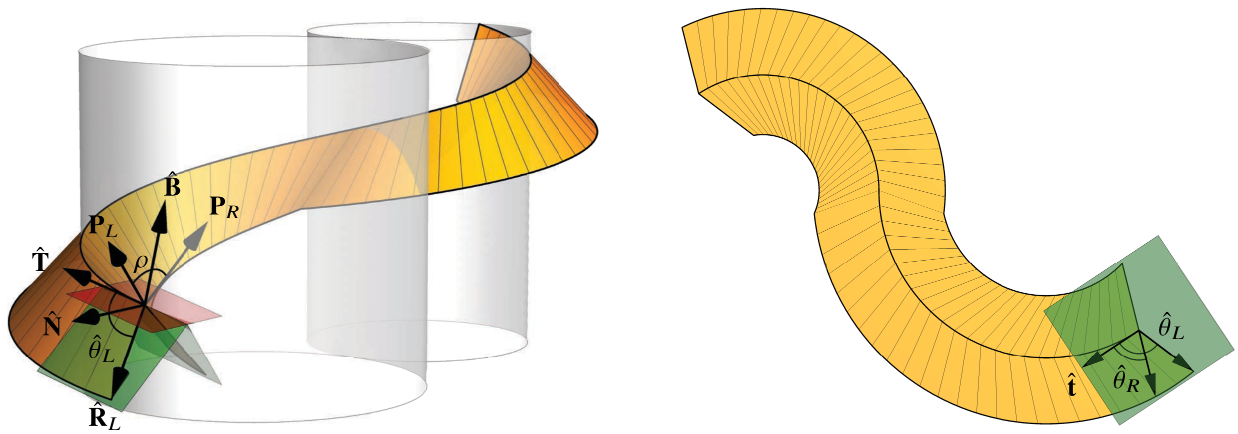


Figure 4: *Illustration of a combination of two helices joined such that tangent and osculating plane coincide (left). We show the development of the curved crease (right).*

Folding motion

As the described shapes have constant fold angle, we can construct a family of intermediate shapes by folding partially while fixing the rule lines on the surface. In the literature, this is also referred to as rigid ruling folding. See Figure 3 for an illustration.

Mathematical Description of Folding with Helical Crease Curves

In the following discussion, we follow the notation of [2]. We consider arc length parametrized helices

$$\mathbf{X}(s) = R\left(\cos\frac{s}{\sqrt{1+h^2}R}, \delta\sin\frac{s}{\sqrt{1+h^2}R}, \frac{hs}{\sqrt{1+h^2}R}\right),$$

that are twisted to the left if $\delta = 1$ and to the right if $\delta = -1$. The accompanying Frenet frame of $\mathbf{X}(s)$ reads

$$\begin{aligned}
\mathbf{T}(s) &= \frac{1}{\sqrt{1+h^2}}\left(-\sin\frac{s}{\sqrt{1+h^2}R}, \delta\cos\frac{s}{\sqrt{1+h^2}R}, h\right),\\
\mathbf{N}(s) &= \left(-\cos\frac{s}{\sqrt{1+h^2}R}, -\delta\sin\frac{s}{\sqrt{1+h^2}R}, 0\right),\\
\mathbf{B}(s) &= \frac{1}{\sqrt{1+h^2}}\left(h\delta\sin\frac{s}{\sqrt{1+h^2}R}, -h\cos\frac{s}{\sqrt{1+h^2}R}, \delta\right).
\end{aligned}$$

We conclude that two helices with vertical axes can only share an osculating plane, if they have the same slope. As the normal and bi-normal vectors change their direction with changing signs of δ, we define the left normal and left bi-normal as $\hat{\mathbf{N}}(t) = \delta\mathbf{N}(t)$ and $\hat{\mathbf{B}}(t) = \delta\mathbf{B}(t)$. The signed curvature and torsion of the space curve are

$$\hat{K} = \frac{d}{ds}\mathbf{T}(s) = \frac{\delta}{R(1+h^2)} \quad \text{and} \quad \tau = -\frac{d}{ds}\hat{\mathbf{B}}(s)\cdot\hat{\mathbf{N}}(s) = \frac{h\delta}{R(1+h^2)}.$$

Intuitively, the curvature is positive when the curve turns to the left and negative when it turns right.

We do a similar setup for the developed curve, which we would like to be an arc length parametrized circular arc with radius r, i.e.,

$$\mathbf{x}(s) = r\left(\cos\frac{s}{r}, \delta\sin\frac{s}{r}\right),$$

with the accompanying Frenet frame composed of the tangent vector $\mathbf{t}(s) = \frac{d}{ds}\mathbf{x}(s)$ and normal vector $\mathbf{n}(s) = -\frac{1}{r}\mathbf{x}(s)$. The left normal and signed developed curvature read $\hat{\mathbf{n}}(s) = \delta\mathbf{n}(s)$ and $\hat{k} = \frac{\delta}{r}$.

As our goal is to glue a surface with a circular opening to a helix, we need to ensure that the geodesic curvature of the curve with respect to the glued surface equals the developed curvature. Following the notation of [2], let $\rho \in [0, \pi]$ be two times the angle between the adjacent developable surface normal $\mathbf{P}_i(s)$, which is constant along a ruling, and $\hat{\mathbf{B}}(s)$. Then, the geodesic curvature of the Darboux frame of $\mathbf{X}(s)$ with normal vector $\mathbf{P}_i(s)$ reads $\kappa_g = \hat{K}\cos\frac{\rho}{2}$. Setting κ_g to be equal to the developed curvature $\hat{k}$ yields the relationship

$$\cos\frac{\rho}{2} = \frac{\hat{k}}{\hat{K}} = \frac{R}{r}(1+h^2). \tag{2}$$

This confirms that ρ is constant. Furthermore, for $\rho = \pi$, i.e., the complete fold, the crease curve becomes a straight line.

The two admissible normal vectors that correspond to the surfaces to the left and right of the crease read

$$\mathbf{P}_i(s) = \cos\frac{\rho}{2}\hat{\mathbf{B}}(s) + \sigma_i\sin\frac{\rho}{2}\hat{\mathbf{N}}(s) \quad \text{where} \quad i \in \{L, R\},\ \sigma_R = 1 \text{ and } \sigma_L = -1.$$

We make the assumption that the ruling direction can be written as

$$\mathbf{R}_i(s) = \cos\hat{\theta}_i\mathbf{T}(s) + \sin\hat{\theta}_i(\cos\frac{\rho}{2}\,\hat{\mathbf{N}}(s) + \sigma_i\sin\frac{\rho}{2}\,\hat{\mathbf{B}}(s)),$$

where $\hat{\theta}_i$ denotes the *ruling angle*. This angle is determined by the developability condition of the adjacent surfaces and reads

$$-\cot\hat{\theta}_L = \cot\hat{\theta}_R = \frac{h}{\sin\frac{\rho}{2}}.$$

This confirms that two helices with the same slope and fold angle can be joined. In particular, this also holds for straight segments that are equipped with the same fold angle and the same "ruling angle".

The oriented distance v to the edge of regression $\mathbf{X}(s) + v\mathbf{R}_i(s)$ of developable surfaces is characterized by the linear dependence of the direction vector $\mathbf{R}_i$ and tangent of the edge of regression, $\frac{d}{ds}\,(\mathbf{X}(s) + v\mathbf{R}_i(s))$. For helical developables, this distance reads

$$v = \frac{2(1+h^2)R\tan\frac{\rho}{2}}{\sqrt{2(1-\cos\rho)+4h^2}}.$$

As long $\rho > 0$ and ($R > 0$), we can attach surfaces of non-zero width. If we take the offset in direction the rulings, it will in general not be tangent continuous for not perpendicular rulings, see Figure 4. In our design, we therefore make sufficiently small orthogonal offsets.

Implementation

We now briefly discuss the main ideas behind the implementation.

Step 1: Define a curve

Firstly, the user is requested to design the initial shape for the central curve of the design. As our method requires a curve that is composed of tangent continuously joined circular arcs and straight line segments, this can be done by interpolating user-specified points: The first three points $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2$ define the first arc or line $\mathbf{X}_1$. Having constructed the curve for the points $P_1, \ldots, P_i$, the next segment is the uniquely determined line or circular arc that shares the tangent direction with the previous segment at P_i and passes through P_{i+1}.

Step 2: Define the parameters

In the next step, the user can adjust the fold angle and slope of the helix. This defines the maximal possible thickness of the strip along the central curve before the boundary curve hits a singular curve on a surface. If the user chooses to remain with $h = 0$, the piece-wise conical surfaces can be pleated.

Step 3: Compute the development

The inserted parameters define the curvatures and thus radii of the developed circles by Equation (2). By tangent continuous combination and appropriate offsets, this then fully defines the development. We do not detect whether collisions occur in the developed curve.

Examples and Font

Figure 5 shows the folded curves that we designed by hand for the font. In addition, the computed unfolded (developed) curves, according to a constant fold angle of $\pi/3$, are displayed. The latter makes a challenging puzzle font: folding along the specified crease produces the letter, but in most cases, that letter is not obvious.

Figure 6 shows a real-world construction of "Bridges" in the proposed font, folded from four strips of polypropylene. We taped a wire to the back side of the crease line to hold the folding in place.

References

[1] E. D. Demaine, M. L. Demaine, D. A. Huffman, D. Koschitz, and T. Tachi. "Characterization of curved creases and rulings: Design and analysis of lens tessellations." *Origami^6: Proceedings of the 6th International Meeting on Origami in Science, Mathematics and Education*, vol. 1, 2014, pp. 209–230.

[2] E. D. Demaine, M. L. Demaine, D. A. Huffman, D. Koschitz, and T. Tachi. "Conic crease patterns with reflecting rule lines." *Origami^7: Proceedings of the 7th International Meeting on Origami in Science, Mathematics and Education*, vol. 2, 2018, pp. 573–590.

[3] E. D. Demaine, M. L. Demaine and K. Mundilova. "Curved Crease Designer." http://erikdemaine.org/fonts/curvedcrease/design.html, 2018.

[4] D. Fuchs and S. Tabachnikov. "More on paperfolding." *The American Mathematical Monthly*, vol. 106, 1992, pp. 27–35.

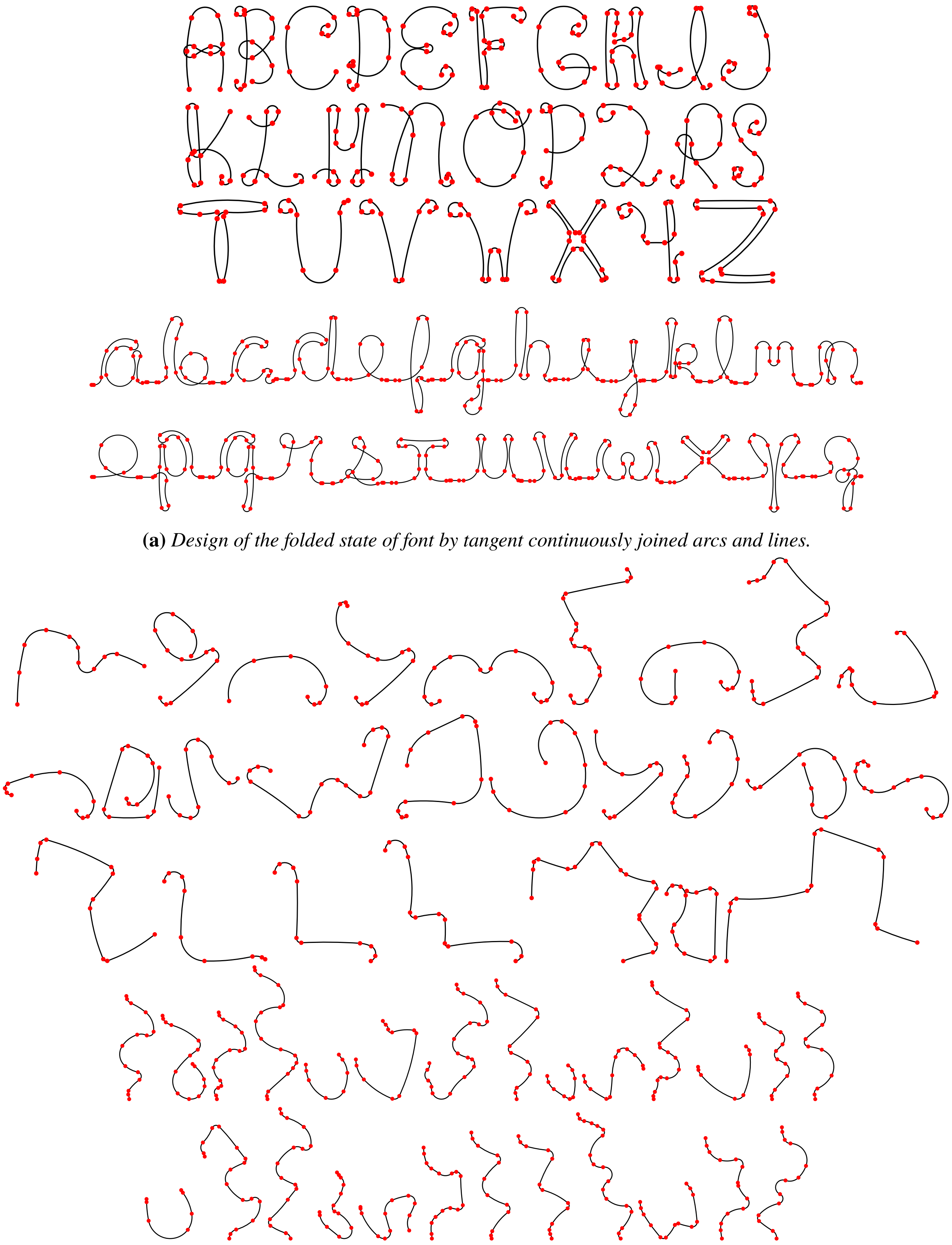
[5] J. Mitani and T. Igarashi. "Interactive design of planar curved folding by reflection." *Pacific Graphics*, 2011, pp. 77–81.

[6] M. Rabinovich, T. Hoffmann and O. Sorkine-Hornung. "Modeling curved folding with freeform deformations." *ACM Trans. Graph.*, vol. 38, 2019, Article 170.

[7] C. Jiang, K. Mundilova, F. Rist, J. Wallner, and H. Pottmann. "Curve-pleated structures." *ACM Trans. Graph.*, vol. 38, 2019, Article 169.

[8] Design is Fine. History is Mine. "Josef Albers, paper folding at the Black Mountain College, 1940s." Blog post, 2014. https://www.design-is-fine.org/post/97720883189/josef-albers-paper-folding-at-the-black-mountain.

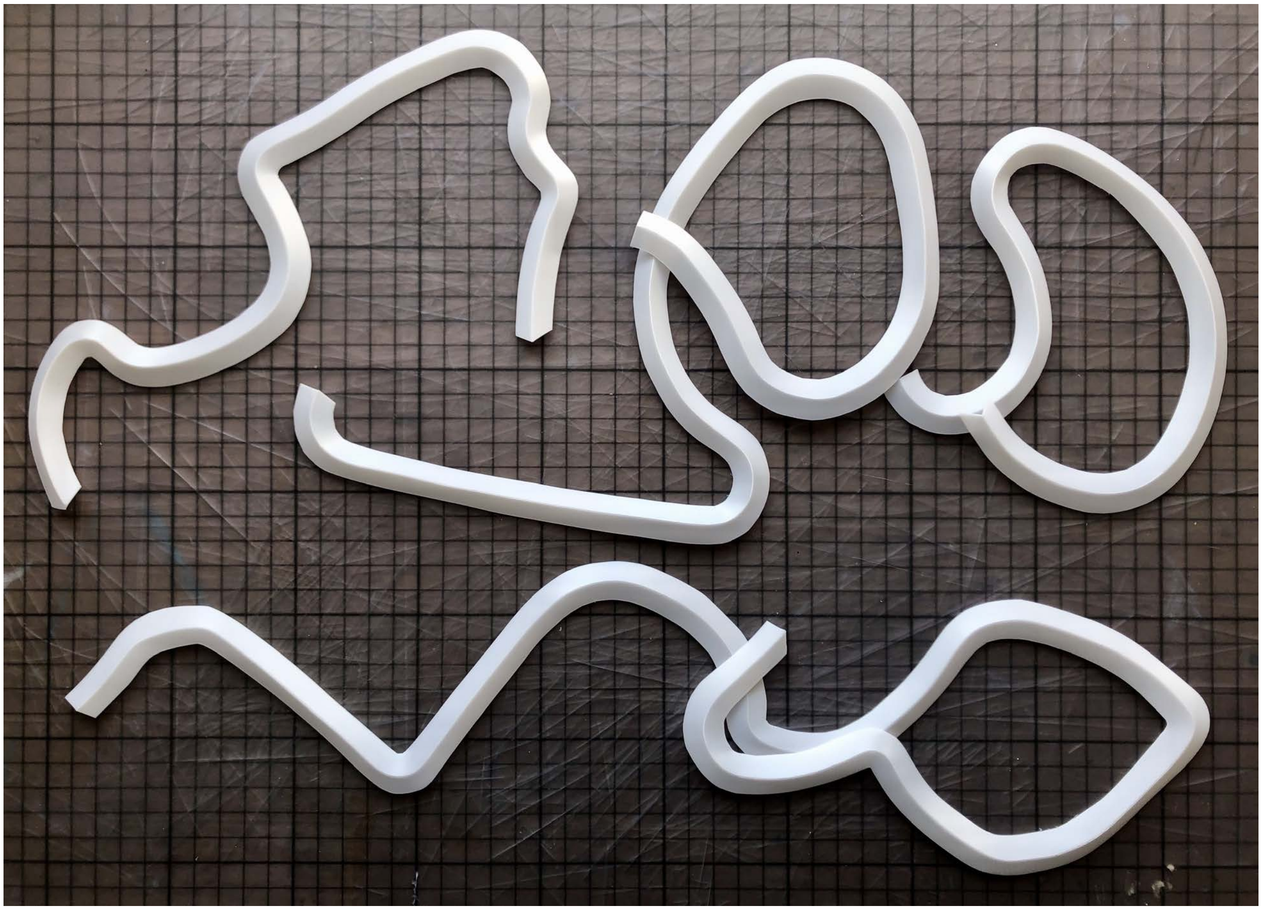
[9] J. Mitani. *Curved-Folding Origami Design*. CRC Press, 2019.



(a) *Design of the folded state of font by tangent continuously joined arcs and lines.*

(b) *Developments of the above crease curves according to the fold angle* $\rho = \frac{\pi}{3}$.

Figure 5: *Upper- and lower-case fonts.*



(a) *Unfolded text.*

(b) *Folded text.*

Figure 6: *Application of the proposed font to the text "Bridges".*