Computational Balloon Twisting: The Theory of Balloon Polyhedra

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Abstract

This paper builds a general mathematical and algorithmic theory for balloon-twisting structures by modeling their underlying edge skeleta, evolving classic balloon animals into the new world of balloon polyhedra.

What if Euler were a clown?

1 Overview

Balloon twisting (or *balloon modeling*) is a form of sculpture rooted in the magic community starting in the 1930s [1]. Modern balloon twisters gather at the annual *Twist & Shout* convention[1] and are the subject of an excellent documentary [5]. In this paper, we investigate the geometric and algorithmic nature inherent in this art form, founding the new field of *computational balloon twisting*. We use this perspective to design a new class of balloon-twisted sculpture called *balloon polyhedra*.

We begin with the basics of practical balloon twisting (Section 2) and their mathematical idealizations called "bloons" (Section 3). Then we consider the mathematics of three such models in turn: simple twisting (Section 4), pop twisting (Section 5), and equalizing bloon lengths (Section 6). Finally, we find optimal constructions for Platonic and Archimedean solids (Section 7).

In addition to artistic applications, computational balloon twisting has potential applications to building architectural structures. Our results suggest that a long, low-pressure tube (called an *air beam* in architecture) enables the temporary construction of inflatable shelters, domes, and many other polyhedral structures, which can be later reconfigured into different shapes and re-used at different sites.

2 Balloon Basics

The majority of balloon twisting starts from a long, narrow balloon, the most common being the "260" which measures 2 inches in diameter and 60 inches in length when fully inflated. Normally the balloonist only partially inflates such a balloon, however, leaving one end deflated as in Figure 3(a). This deflated end leaves room for the air to spread out when *twisting* the balloon along a circular cross-section, forming a *vertex* as in Figure 3(b). The vertex holds its shape if wrapped around another vertex, as in Figure 3(c). The figure shows the vertex coming from another balloon, but it could just as well come from another part of the same balloon, as in the middle of Figure 3(d). Indeed, one theme in balloon twisting is designing complex figures (often animals) from a single balloon, and in this paper we often aim for this goal or for minimizing the number of balloons. Vertex joints can also be bent, similar to joints in a linkage, and will hold their shape if the linkage forces them to remain bent by a nontrivial angle, as on the right of Figure 3(d).

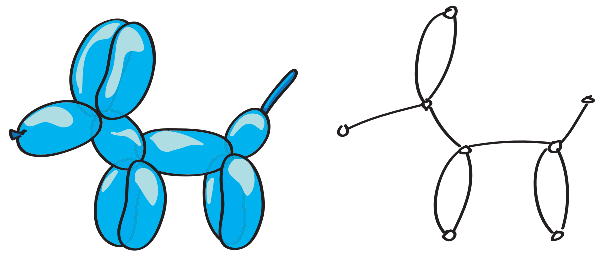

Figure 1: Classic dog (one balloon).

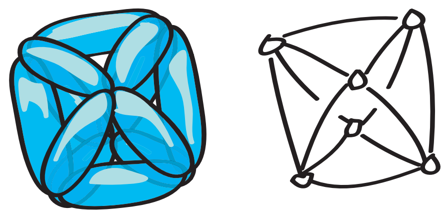

Figure 2: Octahedron (one balloon).

3 Twistable Tangles: Bloon Models

Inflated balloon segments and their twisted end vertices naturally form a graph. Our central problem is to determine which graphs are *twistable* under a variety of abstract models of physical balloons, which we refer to as "bloons" for contrast.

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[1] http://www.balloonconvention.com/

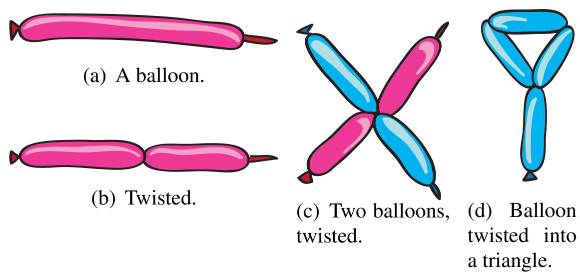


Figure 3: Twisting balloons.

In general, a *bloon* is a segment which can be *twisted* at arbitrary points to form *vertices* at which the bloon can be bent like a hinge. The endpoints of a bloon are also vertices. Two vertices can be *tied* to form permanent point connections. A twisted bloon is *stable* if every vertex is either tied to another vertex or held at a nonzero bending angle.

We distinguish two main models of bloon twisting:

1. *(Simple) twisting*: Every subsegment of a bloon between two vertices form an edge in the associated graph, representing an inflated portion of a balloon.
2. *Pop twisting*: Some subsegments of a bloon between two vertices can be marked as *deflated*, causing them not to appear in the associated graph. Such deflated segments can be achieved with physical balloons by squeezing the air down the balloon or by popping a segment between two existing vertices (a practice common in balloon twisting, though requiring some care and skill).

Two other parameters shape the model:

3. *Number of bloons*: In general we allow structures consisting of any number k of bloons. Of particular interest are the case $k = 1$ and minimizing the number of bloons. A graph has *bloon number* k if it can be simply twisted from k bloons and no fewer.
4. *Bloon lengths*: For multibloon structures, we prefer the bloons to have the same or similar lengths. In particular, this constraint helps us avoid the need for extremely long balloons (which are difficult to obtain). An *ℓ-bloon* is a bloon of length ℓ. We often consider graphs whose edges have unit length, and hence particular cases like *doubloons* ($\ell = 2$) and *demidoubloons* ($\ell = 1$) are of interest.

4 Euler Outgrowth: Bloon Number

Simple twisting of a single bloon naturally forms an Eulerian tour of the constructed graph. Thus single-bloon graphs must have vertices of even degree, except possibly for two odd degrees, and must be connected. Indeed, such graphs are always twistable:

Theorem 1 *A graph has bloon number* 1 *if and only if the graph is Eulerian.*

More interesting from a technical standpoint is the case of k bloons (of arbitrary lengths). Here we can exactly characterize bloon number:

Theorem 2 *A graph with $o > 0$ odd vertices has bloon number $o/2$.*

Proof. Every odd-degree vertex must have an odd number of bloon ends, and each bloon has only two ends, so $o/2$ bloons are necessary. To see that $o/2$ bloons suffice, consider adding $o/2$ edges connecting the odd-degree vertices in pairs. (Recall that every graph has an even number of odd-degree vertices.) The resulting graph has all even degrees and hence an Euler tour. Removing the $o/2$ added edges from the tour results in $o/2$ paths, which are the desired bloons. □

5 Chinese Connection: Pop Twisting

Pop twisting is of course the more general model: it allows building any graph (without straight degree-2 vertices) from a single bloon. In this context, the natural objective is to minimize the total deflated length of the bloon, or equivalently, the total length of the bloon.

This problem is similar to the *Chinese Postman Problem*: given a graph, find a tour of minimum length that visits all edges. This problem has a classic polynomial-time solution based on adding to the graph a minimum-cost perfect matching of the complete graph K_o on the o odd-degree vertices, resulting in the cheapest Eulerian supergraph. The costs in the complete graph can be defined by shortest paths in the graph (for hiding deflated segments against inflated segments), or to include shortcuts available to the bloon in 3D.

The difference is that a pop twisting of a polyhedron requires a path, while the Chinese postman finds the optimal tour (cycle). To find the optimal path, we instead add the minimum-cost $(o/2 - 1)$-edge matching in K_o, leaving exactly two odd vertices. More generally, if we are given k bloons instead of one, we can add a minimum-cost $(o/2 - k)$-edge matching, leaving exactly $2k$ odd vertices; by Theorem 2, the resulting graph can be traversed by k paths. Such a matching can be computed as a minimum-cost maximum flow in the complete bipartite graph $K_{o,o}$, with edge costs defined as in K_o, together with a source of capacity $o/2 - k$ attached to one side of the bipartition via edges of capacity 1, and a sink attached to the other side of the bipartition via edges of capacity 1.

Theorem 3 *There is a polynomial-time algorithm that, given a graph and a desired $k \geq 1$, finds the k bloons of minimum total length that pop-twist the graph.*

6 Length Limitations: Holyer's Problem

Given a graph simply twistable from k bloons, how similar in length can the k bloons be? In particular, when can the lengths all be identical? We can specialize further to obtain a clean combinatorial problem by supposing graph edges all have unit length, as in regular polyhedra, and the bloons have integer length ℓ. What graphs can be simply twisted from ℓ-bloons?

This problem is closely related to *Holyer's problem*: decide whether the edges of a graph can be decomposed into copies of a fixed graph H. In 1981, Holyer [8] conjectured that this problem is NP-complete if H has at least three edges. This conjecture turns out to be correct when H is connected. In fact, the problem is NP-complete if H has a connected component consisting of at least three edges [3], and otherwise it can be solved in polynomial time [2]. Of particular relevance is an old result that every graph with an even number of edges can be decomposed into length-2 paths [11]:

Theorem 4 *Every graph with unit edge lengths can be twisted from doubloons and possibly one demidoubloon (when the graph has an odd number of edges).*

For $\ell > 2$, however, there is a discrepancy between Holyer's problem and simply twisting from ℓ-bloons. On the one hand, each ℓ-bloon can be twisted into any Eulerian graph on ℓ edges. On the other hand, Holyer's problem assumes all bloons form the same such graph, e.g., a path of ℓ edges or a cycle of ℓ edges. Therefore the known NP-hardness for Holyer's problem beyond two edges does not immediately imply NP-hardness for simple twisting beyond doubloons. Fortunately, one NP-hardness proof for Holyer's problem also establishes NP-hardness of simple twisting:

Theorem 5 *It is NP-complete to decide whether a planar bipartite graph with unit edge lengths can be simply twisted from ℓ-bloons.*

Proof. Dyer and Frieze [4, Theorem 3.4] prove NP-hardness of Holyer's problem when the graph to decompose is planar and bipartite and the pattern graph H is a path of length $\ell > 2$. Their reduction has the additional feature that all cycles have length larger than ℓ, and hence no ℓ-bloon could form a structure other than a path of length ℓ. □

We can specialize even further and still obtain NP-hardness. Theorem 2 characterizes the fewest bloons required for simple twisting. When can these fewest bloons have the same length?

Theorem 6 *It is strongly NP-complete to decide whether a planar 3-connected graph with o odd vertices can be simply twisted from $o/2$ equal-length bloons.*

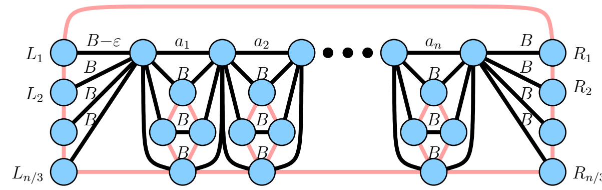


Figure 4: NP-hardness of using the fewest possible equal balloons.

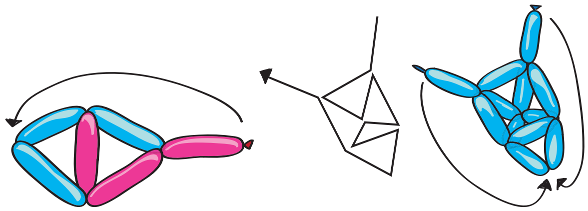
Proof. Figure 4 shows a reduction from 3-partition: given integers $a_1, a_2, \ldots, a_n$, partition into triples of equal sum. The light portion of the graph just makes the graph 3-connected. The dark portion consists of $n/3$ odd-degree vertices on the left, $L_1, L_2, \ldots, L_{n/3}$, and $n/3$ odd-degree vertices on the right, $R_1, R_2, \ldots, R_{n/3}$. All other vertices will have even degree. We can imagine building $n/3$ paths between corresponding L_i and R_i, for $1 \le i \le n/3$, and then pinching these paths together at $n+1$ meeting points. Then a left-to-right path has a choice at each meeting point of which path to follow. Exactly one path can follow an edge of length a_i; the others follows paths of total length B. Here $B > a_1 + a_2 + \cdots + a_n$. Thus each path must visit an equal number of B's, i.e., $n - n/3 + 2$ of them. The path including L_1 has a special edge of length ε less. Here $\varepsilon < \min\{a_1, a_2, \ldots, a_n\}$ and the total length of the light portion of the graph is ε. Thus only this path can visit light edges, and must visit all. Therefore the graph can be twisted by $n/3$ equal-length bloons if and only if the 3-partition instance has a solution. □

By suitable scaling, we can make all edge lengths integers, and then subdivide edges into unit lengths. It seems somewhat difficult, however, to make the graph 3-connected by adding a suitable light Eulerian graph.

Some positive results are known for special cases of Holyer's problem. For example, every 4-regular connected graph whose number of edges is divisible by 3 can be decomposed into paths of length 3, and hence simply twisted from tribloons [7]. The same decomposition and twisting results hold for triangulated (maximal) planar graphs with at least four vertices [6]. It is conjectured that every simple planar 2-edge-connected graph whose number of edges is divisible by 3 can be decomposed into paths and cycles of length 3, and hence simply twisted from tribloons [9]. See also [10]. But relatively few results are known for sizes larger than 3.

7 Polyhedral Projects: Balloon Polyhedra

In contrast to the hardness result of Theorem 6, we show that every Platonic and Archimedean solid can be twisted using the bloon number of bloons, $o/2$, all of equal length. Furthermore, these solids can be twisted



(a) Tetrahedron construction. (b) Octahedron construction.

Figure 5: Constructing two Platonic solids.

so that the component bloon units are all isomorphic and arranged in a symmetric manner. This property makes these polyhedra particularly easy to construct, and lends itself well to color patterns. See Figure 6.

Figure 5 shows how to construct two Platonic solids: the tetrahedron and octahedron. We consider the icosahedron below because it can also be viewed as a snub tetrahedron. The cube and dodecahedron are both possible with tribloons, and together with the tetrahedron are special in that the bloon units can have only dihedral symmetry. In contrast, the icosahedron construction has pyrite symmetry, while the Archimedean constructions below (and the octahedron) have the same symmetry group as the original polyhedron.

The Archimedean solids can be categorized into three different groups for our purposes: Eulerian, truncated, and snub. The Eulerian case is of course trivial. For truncated polyhedra, the optimal bloons are tribloons, because each original edge truncates to create two vertices and three edges, yielding a $2:3$ vertex-edge ratio. The tribloons can be embedded as Zs (or Ss, as the result is chiral), where each center edge aligns with an edge of the original (untruncated) polyhedron, with the arms bending to form the truncated faces. The snub polyhedra (including the icosahedron) can be made from a common unit, namely, a quintibloon twisting into the shape of two triangles sharing an edge.

Of course, not all polyhedra can be made from the bloon number of bloons, $o/2$, of equal length. The pentagonal pyramid is a simple example. It has ten edges and six vertices, all of odd degree, yielding a bloon number of 3. Unfortunately, 10 is not divisible by 3, so one bloon must have length 4. In the realm of polyhedra with icosahedral symmetry, the simplest counterexample is the rhombic triacontahedron, with 60 edges and 32 vertices, which again do not divide evenly. It remains open whether any polyhedron fails to have a twisting from a bloon number of equal-length bloons but not by virtue of indivisibility. It also remains open whether some symmetric polyhedron can be twisted only from nonidentical units or only from units arranged asymmetrically.

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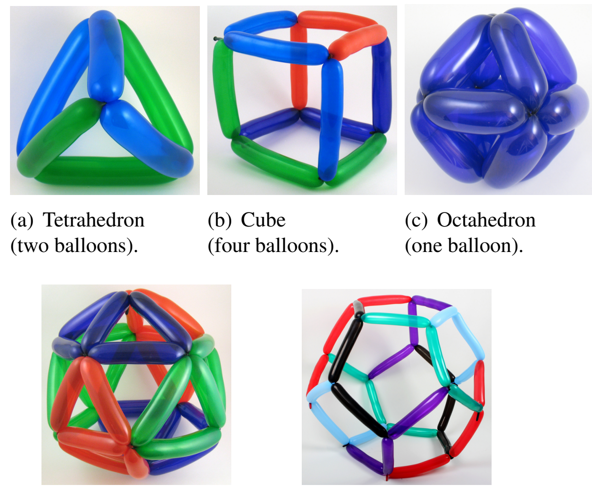

(a) Tetrahedron (two balloons). (b) Cube (four balloons). (c) Octahedron (one balloon). (d) Icosahedron (six balloons). (e) Dodecahedron (six balloons).

Figure 6: Balloon polyhedra.

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