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Worst-case Comparison of Valid Inequalities for the TSP

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Abstract

We consider most of the known classes of valid inequalities for the graphical travelling salesman polyhedron and compute the worst-case improvement resulting from their addition to the subtour polyhedron. For example, we show that the comb inequalities cannot improve the subtour bound by a factor greater than $\frac{10}{9}$. The corresponding factor for the class of clique tree inequalities is $\frac{8}{7}$, while it is $\frac{4}{3}$ for the path configuration inequalities.

1 Introduction

In the last decade, strong cutting-plane methods have successfully solved larger and larger symmetric travelling salesman problem (TSP) instances to optimality. In the early nineties, Padberg and Rinaldi [27] have solved optimally an instance with $2,392$ cities, and more recently Applegate et al. [1]) have solved optimally instances with $3,038$, $4,461$ and even $7,397$ cities. These cutting-plane methods are based on an extensive investigation of the facets of the symmetric travelling salesman polytope ($STSP$) and its relatives. Many classes of facet-defining valid inequalities are now known for $STSP$, see Chvátal [5], Grötschel and Padberg [15], Grötschel [12], Padberg and Hong [28], Cornuéjols et al. [7], Grötschel and Pulleyblank [17], Fleischmann [9], Boyd and Cunningham [3], Naddef and Rinaldi [23, 24] and Naddef [22].

The success of cutting-plane methods for the TSP is however not fully understood. In particular, it is not clear which classes of inequalities are most or least useful in solving

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TSP instances. For obvious reasons of efficiency, the choice of the inequalities to use in a cutting-plane algorithm is typically dictated by whether efficient exact or heuristic algorithms are known for the corresponding separation problem. For this reason, cutting-plane approaches tend to use, in addition to the subtour elimination constraints, the comb inequalities or the clique tree inequalities, see Grötschel [13], Padberg and Hong [28], Crowder and Padberg [8], Grötschel and Holland [14], Padberg and Rinaldi [26, 27], Jünger et al. [19] and Applegate et al. [1]. An exception is a recent implementation of Clochard and Naddef [6] which is based on the path inequalities. The motivation of this paper is best expressed by the following quote from Naddef and Rinaldi [24]:

> For which ... inequalities ... would it be worth investing research time to find good separation algorithms, and computation time to run them in a cutting-plane algorithm? An answer to this question would be easier if there was some indicator that could be used to compare two classes of inequalities, with respect to their potential effectiveness in a polyhedral cutting-plane algorithm.

In this paper, we describe such an indicator by evaluating the worst-case improvement resulting from the addition of many classes of valid inequalities to the subtour polyhedron.

For the reader familiar with polyhedral studies of the TSP, Table 1 gives a preview of the results. The column "Strength" gives for various classes of inequalities the worst-case ratio between the values obtained by optimizing over the subtour polyhedron plus an additional class of inequalities and over the subtour polyhedron itself. The inequalities are assumed to be in graphical form and the worst-case is taken over all nonnegative cost functions. In some sense, "worst-case" has to be understood as best-case, since the larger the improvement the better it is. The most notable entries in the table are the fact that clique tree inequalities cannot improve the subtour polytope bound by more than a factor of $8/7$, while the path configuration inequalities improve it in the worst-case by $4/3$.

The paper is structured as follows. Section 2 reviews some well-known polyhedral results for the TSP. In Section 3, we formally describe a measure to compare two polytopes and show how it can be used to compare various relaxations of the TSP. The results summarized in Table 1 are established in Section 4 and interpreted in Section 5. Finally, we conclude with some remarks in Section 6.

2 Background material

We assume a basic knowledge of polyhedral combinatorics, especially related to the travelling salesman problem. The reader is referred to Grötschel and Padberg [16] or Pulleyblank [29] for a very thorough introduction.

Table 1: Strength of TSP inequalities.

Class of inequalities		Strength	Ref.
Comb	with t teeth	$\frac{3t+1}{3t}$	Cor. 4
	in general	$\frac{10}{9}$	Cor. 4
Clique tree	with h handles and t teeth	$\frac{3t+2h-1}{3t+h-1}$	Thm. 3
	with h handles	$\frac{8h+2}{7h+2}$	Cor. 4
	in general	$\frac{8}{7}$	Cor. 4
Path configuration	with k paths	$\frac{k+1}{k}$	Thm. 5
	in general	$\frac{4}{3}$	Thm. 5
Crown	with parameter k	$\frac{6k(k-1)-1}{6k(k-1)-k}$	Thm. 6
	in general	$\frac{11}{10}$	Thm. 6
Hypohamiltonian	on k vertices	$\leq \frac{k+1}{k}$	Thm. 7
	in general	$\frac{11}{10}$	Thm. 7

The *symmetric travelling salesman polytope*, $STSP$, is defined as the convex hull of incidence vectors of Hamiltonian cycles. Optimizing any cost function over $STSP$ is equivalent to finding an Hamiltonian cycle of minimum cost. There are several drawbacks associated with the polytope $STSP$. First, it is not full-dimensional and this makes its study more difficult. Also, in our worst-case analysis, it will be crucial to consider any nonnegative cost function, while one typically considers instances of the TSP for which the cost function satisfies the triangle inequality. To avoid both drawbacks, we will allow multiple visits of a vertex and consider the problem of finding an Eulerian subgraph (or multigraph) of minimum cost and its associated polyhedron. Given a graph $G = (V, E)$, an Eulerian sub(multi)graph[1] is a collection of edges of E (with possible repetition) such that the degree of every vertex is even and the subgraph is connected. For any cost function, the cost of the minimum Eulerian subgraph is equal to the value of the optimum Hamiltonian cycle with respect to the shortest path costs. The convex hull of the incidence vectors of Eulerian subgraphs is called the *graphical travelling salesman polyhedron*, $GTSP$. This polyhedron was introduced by Cornuéjols et al. [7] and Fleischmann [9] and was further investigated by Naddef and Rinaldi [25, 23, 24], Naddef [22] and Fonlupt and Naddef [10]. $GTSP$ is full-dimensional since it is of blocking type, i.e. if $x \in GTSP$ and $y \geq x$ then $y \in GTSP$.

Many classes of facet-defining valid inequalities are known for $GTSP$. The simplest is the class of *subtour elimination constraints* (also called *loop* constraints in [16] or *cocycle* inequalities in [25]): $x(\delta(S)) \geq 2$ for any $S \subset V$, where $x(F) = \sum_{e \in F} x_e$ and $\delta(S)$ represents the coboundary of S defined as $\{e = (i,j) : |S \cap \{i,j\}| = 1\}$. Naddef and Rinaldi [25] have shown that any facet-defining inequality for $STSP$ gives rise to a facet-defining inequality for $GTSP$. For this purpose, one needs to convert the valid inequality for $STSP$ into *tight triangular form* by adding or subtracting multiple of the degree constraints $x(\delta(\{i\})) = 2$. An inequality $ax \geq a_0$ is in *tight triangular form* [25] if

1. the nonnegative coefficients a_e satisfy the triangle inequality: $a_{ij} + a_{jk} \geq a_{ik}$ for all i, j, k and

2. for all $j \in V$, there exists $i, k \in V - \{j\}$: $a_{ij} + a_{jk} = a_{ik}$.

Naddef and Rinaldi [25] prove that any non-trivial[2] tight triangular facet-defining inequality for STSP is also a facet-defining valid inequality for $GTSP$. Furthermore, it is easy to see that every facet-defining inequality for $GTSP$, except the nonnegativity constraints and the subtour elimination constraints corresponding to singleton sets, must be in tight

[1] They are also called *tours* or *closed walks*, see [7, 25].

[2] not defining the same inequality as $x_e \geq 0$ for any edge e

triangular form [25]. The most well-known class of inequalities for $STSP$ (and thus also for $GTSP$ when in tight triangular form) is the clique tree inequalities introduced by Grötschel and Pulleyblank [17]. They generalize the comb inequalities of Chvátal [5] and their extensions due to Grötschel and Padberg [15]. Many other classes of inequalities are known for $STSP$ or $GTSP$, including the path inequalities [7], the path tree inequalities [23], the binested inequalities [22], the crown inequalities [24] and the rank inequalities [12].

The simplest relaxation of $GTSP$ consists of the subtour elimination constraints. The corresponding *subtour polyhedron* SP, or subtour elimination polyhedron, is defined as:

$$\begin{array}{rlll} SP & = \{ \quad x \in \mathbb{R}^{|E|}: & x(\delta(S)) \geq 2 & S \subset V \\ & & x_e \geq 0 & e \in E\}. \end{array}$$

From an observation of Cunningham [21] (see Goemans and Bertsimas [11]), it follows that optimizing a nonnegative cost function over SP is equivalent to optimizing the shortest path cost function over the subtour *polytope*, obtained by adding the degree constraints $x(\delta\{i\}) = 2$ for all $i \in V$ to SP. As a result, the value obtained by optimizing over SP is exactly equal to the Held-Karp lower bound [18]. Interpreting their result in terms of GTSP, Wolsey [31] and Shmoys and Williamson [30] have shown that

$$\frac{\text{Min}\{cx : x \in GTSP\}}{\text{Min}\{cx : x \in SP\}} \leq \frac{3}{2},$$

for any nonnegative cost function c. However, this bound does not appear to be tight and, in fact, the following conjecture motivated this study (see Sections 3 and 5).

Conjecture 1 *For any nonnegative cost function c,*

$$\frac{Min\{cx : x \in GTSP\}}{Min\{cx : x \in SP\}} \leq \frac{4}{3}.$$

Instances are known which achieve a bound of $\frac{4}{3}$. One can obtain stronger relaxations of $GTSP$ by adding to SP some of the classes of inequalities mentioned previously.

3 Worst-case comparison of relaxations

If P and Q are polyhedra in $\mathbb{R}^n$ then we say that P is a *relaxation* of Q or Q is a *strengthening* of P if $P \supseteq Q$. Given a relaxation P of a polyhedron Q, we are interested in evaluating how well does P approximate Q. In the context of polyhedral combinatorics, this question arises in two different ways. Under the most classical perspective, Q is a combinatorial optimization polyhedron and one would like to estimate how tight is a given relaxation P of Q. Another possible scenario is that one has a standard relaxation P of some combinatorial optimization polyhedron R and one would like to evaluate the improvement that would result from the addition to P of a certain class of valid inequalities. So, in the latter case, one has two relaxations P and Q of R where Q is a strengthening of P.

How does one compare two relaxations? This question is not well understood and, in fact, one can formulate several notions of "tightness". For a recent investigation on a notion based on the volumes of P and Q, see Lee and Morris [20]. In this paper, we adopt an optimization point-of-view.

Throughout this section, we assume that $P \subseteq \mathbb{R}^n_+$ is a relaxation of a polyhedron $Q \subseteq \mathbb{R}^n_+$ of blocking type. This assumption leads to a very natural notion of tightness. Given $\alpha \geq 1$, we say that P is an α-relaxation of Q or Q is an α-strengthening of P if $Q \supseteq \alpha P = \{\alpha x : x \in P\}$, i.e. Q is a relaxation of αP. Any α-relaxation is also a β-relaxation for any $\beta \geq \alpha$. Also, let $t(P,Q)$ denote the minimum value of α such that P is an α-relaxation of Q. Notice that $t(P,Q) \geq 1$, $t(P,Q) = 1$ iff $P = Q$, and that $t(P,Q)$ could be infinite.

The parameter $t(P,Q)$ can be interpreted from an optimization point-of-view. Let c be any vector in $\mathbb{R}^n_+$. If P is an α-relaxation of Q then clearly

$$\mathrm{Min}\{cx : x \in Q\} \leq \alpha \mathrm{Min}\{cx : x \in P\}.$$

From the separating hyperplane theorem, the polyhedron Q contains the polyhedron αP iff, for any cost vector c,

$$\mathrm{Min}\{cx : x \in Q\} \leq \alpha \mathrm{Min}\{cx : x \in P\}.$$

For a vector c which fails to be nonnegative, this inequality is vacuous and, therefore, the converse statement also holds, as stated in the following lemma.

Lemma 1 *Let P be a relaxation of a polyhedron Q in $\mathbb{R}^n_+$ of blocking type. Then P is an α-relaxation of Q iff, for any nonnegative vector $c \in \mathbb{R}^n$,*

$$Min\{cx : x \in Q\} \leq \alpha Min\{cx : x \in P\}.$$

As a corollary, $t(P,Q)$ is equal to

$$t(P,Q) = \mathrm{Sup}_{c \in \mathbb{R}^n_+} \frac{\mathrm{Min}\{cx : x \in Q\}}{\mathrm{Min}\{cx : x \in P\}}, \tag{1}$$

where, by convention, $\frac{0}{0} = 1$.

The following result gives an alternate characterization of $t(P,Q)$ when a description of Q in terms of linear inequalities is known.

Theorem 2 *Let P be a relaxation of a polyhedron Q in $\mathbb{R}^n_+$ of blocking type where $Q = \{x : a_i x \geq b_i$ for $i = 1, \ldots, m\}$, $a_i \geq 0$, $b_i \geq 0$ for $i = 1, \ldots, m$. Then*

$$t(P, Q) = Max_i \frac{b_i}{d_i},$$

where $d_i = Min\{a_i x : x \in P\}$.

Proof:
From (1), it is clear that

$$t(P, Q) \geq \text{Max}_i \frac{\text{Min}\{a_i x : x \in Q\}}{\text{Min}\{a_i x : x \in P\}} \geq \text{Max}_i \frac{b_i}{d_i}.$$

We therefore need to prove the reverse inequality.

Let c be any nonnegative cost function. By strong duality, we know that

$$\begin{array}{llcll} \text{Min} & cx & = & \text{Max} & b^T y \\ s.t. & Ax \geq b & & s.t. & A^T y = c^T \\ & & & & y \geq 0, \end{array}$$

where ^{T} denotes the transpose. Let y^* be the optimal dual solution of the above program. Then

$$\begin{array}{llcllcl} \text{Min} & cx & = & \text{Min} & (y^*)^T Ax & \geq & \sum_i \left\{ \begin{array}{ll} \text{Min} & a_i x \\ s.t. & x \in P \end{array} \right\} y_i^* = \sum_i d_i y_i^*. \\ s.t. & x \in P & & s.t & x \in P & & \end{array}$$

Hence,

$$\frac{\text{Min}\{cx : x \in Q\}}{\text{Min}\{cx : x \in P\}} \leq \frac{\sum_i b_i y_i^*}{\sum_i d_i y_i^*} = \sum_i \left(\frac{d_i y_i^*}{\sum_j d_j y_j^*} \right) \frac{b_i}{d_i}.$$

Since $d_i \geq 0$ (because $a_i \geq 0$ and $P \subseteq R_n^+$) and $y_i^* \geq 0$, the latter quantity can be interpreted as a convex combination of $\frac{b_i}{d_i}$ and is therefore less or equal to

$$\text{Max}_i \frac{b_i}{d_i}.$$

The result is proved by taking the supremum over all nonnegative cost functions c. $\square$

We should point out that we haven't used the fact that P is a relaxation of Q in the above theorem.

Theorem 2 can be rephrased as follows. To compute $t(P, Q)$, one only needs to consider the cases in which a single inequality of Q is added to P. This motivates the following definition. The *strength* of an inequality $ax \geq b$ with respect to a polyhedron P is defined as

$$\frac{b}{\mathrm{Min}\{ax : x \in P\}}.$$

Theorem 2 implies that $t(P, Q)$ is equal to the maximum strength with respect to P of a facet-defining inequality for Q.

3.1 Comparison of TSP relaxations

In this paper, we evaluate $t(SP, Q)$ for many relaxations Q of $GTSP$, i.e. we evaluate the worst-case improvement due to the use of the relaxation Q instead of the subtour polyhedron SP. From Theorem 2, we know that $t(SP, Q)$ is equal to the maximum strength (with respect to SP) of any facet-defining inequality for Q. In particular, Conjecture 1 which says that $t(SP, GTSP)$ is at most $\frac{4}{3}$, is equivalent to the conjecture that the strength of *any* facet-defining valid inequality for $GTSP$ is at most $\frac{4}{3}$.

Given a valid inequality $ax \geq b$ for $GTSP$, we need to compute $\mathrm{Min}\{ax : x \in SP\}$ in order to evaluate its strength. By strong duality, $\mathrm{Min}\{ax : x \in SP\}$ is equal to

$$(D) \qquad \begin{array}{lll} \mathrm{Max} & 2\sum_{S} y_S & \\ \text{subject to:} & & \\ & \sum_{S : e \in \delta(S)} y_S \leq a_e & e \in E \\ & y_S \geq 0 & S \subset V. \end{array}$$

Notice that if y is a feasible solution to (D) then $\frac{b}{2\sum_S y_S}$ is an *upper bound* on the strength of the inequality $ax \geq b$. In the next section, we derive upper bounds on the strength of many valid inequalities for $GTSP$ by appropriately constructing simple dual feasible solutions. The resulting upper bounds can in most cases be attained.

We should point out that two other notions of "strength" of TSP facet-defining inequalities have been proposed by Naddef and Rinaldi [24]. As a first measure, they suggest to evaluate the number of Hamiltonian cycles on the corresponding facet. Their second measure is somewhat similar to our notion of strength. For an inequality $ax \geq b$ they evaluate the distance between the optimum solution to $\mathrm{Min}\{ax : x \in SP, x(\delta(\{i\})) = 2$ for all $i\}$ and the affine space defined by $\{x : ax = b, x(\delta(\{i\})) = 2$ for all $i\}$. The degree constraints are present in the definition of the affine space to avoid that the measure depends on the representation of the facet in terms of a linear inequality.

4 Strength of TSP inequalities

The results of this section are summarized in Table 1.

4.1 Clique tree inequalities

Clique tree inequalities were discovered by Grötschel and Pulleyblank [17]. A *clique tree* C consists of a collection of subsets of vertices partitioned into *handles* $H_1, \ldots, H_h$ and *teeth* $T_1, \ldots, T_t$ such that:

1. No two teeth intersect,
2. No two handles intersect,
3. Each tooth has at least one vertex not belonging to any handle and
4. The number of teeth that each handle intersect is odd and at least three.
5. The intersection graph of the handles and teeth is a tree.

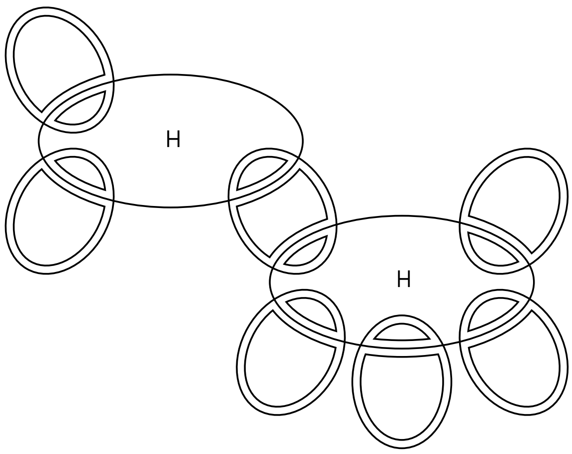

Figure 1: A clique tree with 2 handles (indicated by an H) and 7 teeth. The closed curves correspond to the sets S with $y_S = \frac{1}{2}$ in the proof of Theorem 3.

A clique tree is represented in Figure 1. When the number of handles h is 1, the clique tree is called a *comb*. Combs were introduced by Chvátal [5] for the case with at most one common vertex between a tooth and a handle and by Grötschel and Padberg [15] for the general case. The clique tree inequality (in tight triangular form) is:

$$\sum_{i=1}^{h} x(\delta(H_i)) + \sum_{i=1}^{t} x(\delta(T_i)) \geq 2\sum_{i=1}^{t} t_i + t + 1,$$

where t_i denotes the number of handles intersecting T_i. For simplicity, let $p = \sum_{i=1}^{t} t_i$. Since the intersection graph of the handles and teeth is a tree with $t + h$ vertices and p edges, we have $p = t + h - 1$. Hence, the clique tree inequality can be rewritten as:

$$\sum_{i=1}^{h} x(\delta(H_i)) + \sum_{i=1}^{t} x(\delta(T_i)) \geq 3t + 2h - 1.$$

Clique tree inequalities define facets of $GTSP$ or $STSP$ [17].

Theorem 3 *The strength of a clique tree inequality with h handles and t teeth is at most*

$$\frac{3t + 2h - 1}{3t + h - 1}.$$

From the above theorem, we can derive:

Corollary 4

1. *The strength of a comb inequality with t teeth is at most $\frac{3t+1}{3t}$,*
2. *The strength of a comb inequality is at most $\frac{10}{9}$,*
3. *The strength of a clique tree inequality with h handles is at most $\frac{8h+2}{7h+2}$,*
4. *The strength of a clique tree inequality is at most $\frac{8}{7}$.*

Items 1. and 2. follow from the definition of combs, item 3. from the fact that $t \geq 2h + 1$ (since the number of edges incident to handles in the intersection graph is at least $3h$ and at most $p = t + h - 1$), and item 4. from item 3.

Proof of Theorem 3:

From the discussion in Subsection 3.1, we need to construct a feasible solution for (D) of value $3t + h - 1$. Our dual solution is the following. Let $y_S = \frac{1}{2}$ for any S of the form

$$S = \begin{cases} H_i \cap T_j & 1 \leq i \leq h, 1 \leq j \leq t \text{ and } H_i \cap T_j \neq \emptyset, \\ T_j & 1 \leq j \leq t, \\ T_j - \bigcup_i H_i & 1 \leq j \leq t, \end{cases}$$

and $y_S = 0$ otherwise. (See Figure 1 for an illustration of y.) Notice that, by definition of a clique tree, all these sets are nonempty. Also,

$$2\sum_S y_S = \sum_{i=1}^{t} t_i + 2t = p + 2t = 3t + h - 1.$$

Finally, one can easily verify that y is a dual feasible solution, completing the proof. $\square$

Remark: Using a result of Boyd and Pulleyblank [4], the strength of a clique tree inequality can be shown to be exactly equal to

$$\frac{3t + 2h - 1}{3t + 2h - 2 - npt},$$

where npt denotes the number of nonpendent teeth of the clique tree. A *nonpendent* tooth is a tooth intersecting more than one handle. Indeed, it is easy to derive from Theorem 2.2 of Boyd and Pulleyblank [4] that $Z_{SP} = 3t + 2h - 2 - npt$, where Z_{SP} denotes the optimum value when optimizing over SP. Notice that this is in agreement with Theorem 3 since one can easily show (for example, by induction) that $npt \leq h - 1$. This also shows that our upper bounds on the strength of subclasses of clique tree inequalities described in Theorem 3 and Corollary 4 can be attained.

4.2 Path inequalities

Path inequalities were introduced by Cornuéjols et al. [7]. A k*-path configuration* is defined by an odd integer $k \geq 3$, integers $n_i \geq 2$ for $i = 1, \ldots, k$ and a partition of the vertex set V into $\{A, Z, B^i_j$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n_i\}$ where A and/or Z could be empty. For notational convenience, let $B^i_0 = A$ and $B^i_{n_i+1} = Z$ for $i = 1, \ldots, k$. Let $\delta(S, T)$ be $\delta(S) \cap \delta(T)$. The k-path inequality is $ax \geq a_0$ where

$$a_0 = 1 + \sum_{i=1}^{k} \frac{n_i + 1}{n_i - 1}$$

and

$$a_e = \begin{cases} 1 & e \in \delta(A, Z) \\ \frac{|j-l|}{n_i - 1} & e \in \delta(B^i_j, B^i_l), i = 1, \ldots, k \text{ and } 0 < |j - l| \leq n_i \\ \frac{1}{n_h - 1} + \frac{1}{n_i - 1} + \left|\frac{j-1}{n_h - 1} - \frac{l-1}{n_i - 1}\right| & e \in \delta(B^h_j, B^i_l), h \neq i, j = 1, \ldots, n_h \text{ and } l = 1, \ldots, n_i \\ 0 & \textit{otherwise}. \end{cases}$$

When $n_i = p$ for all i, the k-path inequality is called p-regular and takes a particularly attractive form as can be seen in Figure 2 for the case $k = 3$. The class of 2-regular path inequalities is equivalent to the class of comb inequalities. The k-path inequalities are facet-defining for $GTSP$ [7].

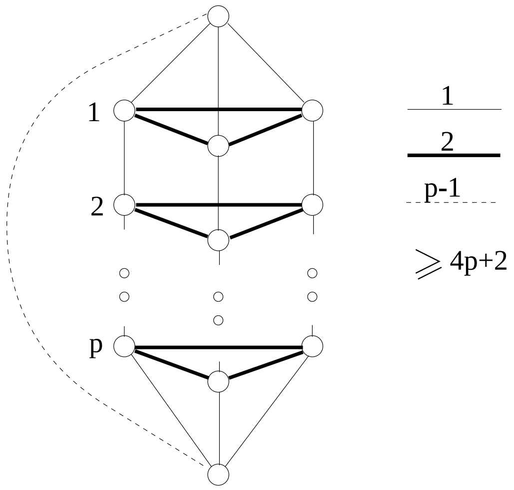

Figure 2: A p-regular 3-path inequality. The coefficient a_e in the inequality $ax \geq 4p+2$ is equal to the shortest path length between the endpoints of e in the above weighted graph.

Theorem 5 *The strength of a path inequality is*

$$\frac{1+\sum_{i=1}^{k} r_i}{\sum_{i=1}^{k} r_i} \leq \frac{k+1}{k} \leq \frac{4}{3},$$

where $r_i = \frac{n_i+1}{n_i-1}$.

Proof:
Let y be defined by (see Figure 3):

$$y_S = \begin{cases} \frac{1}{2}\frac{1}{n_i-1} & S = B_j^i \text{ with } 1 \leq i \leq k \text{ and } 1 \leq j \leq n_i \\ \frac{1}{2}\frac{1}{n_i-1} & S = \bigcup_{j=1}^{n_i} B_j^i \text{ with } 1 \leq i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

One can verify that y is a feasible solution for (D) with value $2\sum_S y_S = \sum_{i=1}^{k} r_i$. Moreover, y is an optimal solution since a primal solution of the same value can be obtained by selecting an edge from each set $\delta(B_j^i, B_{j+1}^i)$ for $i = 1, \ldots, k$ and $j = 0, \ldots, n_i$ as well as

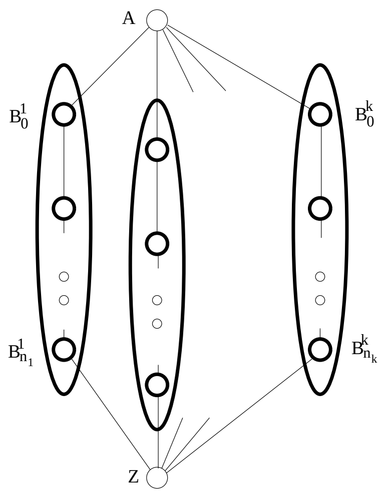

Figure 3: Illustration of the dual solution y in the case of a path configuration inequality. y has a nonzero component only for those sets represented as thick circles or ellipses.

edges from within the B_j^i. Using the arguments of Subsection 3.1, the strength of a path inequality is thus equal to

$$\frac{1+\sum_{i=1}^{k} r_i}{\sum_{i=1}^{k} r_i} = 1 + \frac{1}{\sum_{i=1}^{k} r_i} \leq 1 + \frac{1}{k} \leq \frac{4}{3},$$

where we have used the facts that $r_i \geq 1$ and $k \geq 3$. $\square$

4.3 Crown inequalities

Crown inequalities were recently discovered by Naddef and Rinaldi [24]. Consider a partition of V into $4k$ ($k \geq 2$) vertex sets $U_1, \ldots, U_{4k}$. Given $1 \leq i, j \leq 4k$, let $\langle i,j \rangle$ be defined as the distance on the cycle $\{1, 2, \ldots, 4k, 1\}$, i.e. $\langle i,j \rangle = |i-j|$ if $|i-j| \leq 2k$ and $\langle i,j \rangle = 4k - |i-j|$ otherwise. The crown inequality is the inequality $ax \geq a_0$ where $a_0 = 12k(k-1) - 2$ and

$$a_e = \begin{cases} 4k-6+\langle i,j \rangle & e \in \delta(U_i, U_j), 1 \leq \langle i,j \rangle \leq 2k-1 \\ 2k-2 & e \in \delta(U_i, U_{i+2k}), 1 \leq i \leq 2k \\ 0 & otherwise. \end{cases}$$

The simplest crown inequality is represented in Figure 4. The crown inequalities are facet-defining for $k \geq 2$ [24].

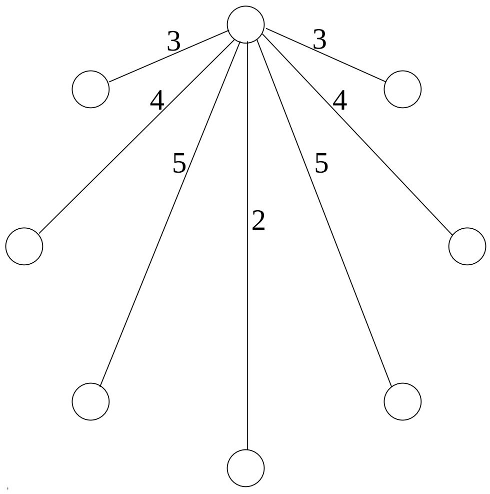

Figure 4: Crown inequality on 8 vertices ($k = 2$). The right-hand-side is 22 and the coefficients of all edges can be obtained by simply rotating the edges of the figure.

Theorem 6 *The crown inequality has strength*

$$\frac{6k(k-1)-1}{6k(k-1)-k} \leq \frac{11}{10}.$$

Proof:
As usual, we claim that the following vector y is a feasible solution to (D) of value $12k(k-1)-2k$:

$$y_S = \begin{cases} k-1 & S = U_i \text{ with } 1 \leq i \leq 4k \\ k-\frac{3}{2} & S = U_i \cup U_{i+2k} \text{ with } 1 \leq i \leq 2k \\ 0 & \text{otherwise.} \end{cases}$$

This vector y is optimal since a primal solution x of the same value can be derived: set $x_e = 1$ for one edge e from each diameter $\delta(U_i, U_{i+2k})$, $x_e = 0.5$ for one edge e from each $\delta(U_i, U_{i+1})$ and also $x_e = 1$ for edges within U_i. □

4.4 Hypohamiltonian inequalities

A *hypohamiltonian graph* is a nonhamiltonian graph such that the deletion of any node yields a hamiltonian graph. The Petersen graph is the smallest hypohamiltonian graph (Sosselier, see Berge [2]). Consider a partition of V into $V_1, \ldots, V_k$ and let H be an edge maximal hypohamiltonian graph with vertex set $\{1, \ldots, k\}$. The resulting hypohamiltonian inequality is $ax \geq a_0$ where $a_o = k + 1$ and

$$a_e = \begin{cases} 1 & e \in \delta(V_i, V_j), i \neq j \text{ and } (i,j) \in H \\ 2 & e \in \delta(V_i, V_j), i \neq j \text{ and } (i,j) \notin H \\ 0 & otherwise. \end{cases}$$

The hypohamiltonian inequalities were shown to be facet-defining for $GTSP$ by Cornuéjols et al. [7]. For related results for $STSP$, see Grötschel and Padberg [16].

Theorem 7 *The strength of a hypohamiltonian inequality corresponding to a hypohamiltonian graph on k vertices is at most*

$$\frac{k+1}{k} \leq \frac{11}{10}.$$

Proof:
Let $y_S = \frac{1}{2}$ for any S of the form V_i and $y_S = 0$ otherwise. Clearly, y is a feasible solution to (D) of value k. Hence, an upper bound on the strength is $\frac{k+1}{k}$. This is at most $\frac{11}{10}$ since the smallest hypohamiltonian graph is the Petersen graph (on 10 vertices). □

4.5 Miscellany

As we have seen, it is often child's play to compute the strength (or at least an upper bound on the strength) of an inequality for $GTSP$. The reader can experiment with his own favorite class of inequalities. As a last example, one can easily show that a planted clique tree inequality as defined by Naddef and Rinaldi [23] has strength upper bounded by

$$\frac{4t + 2h}{4t + h - 1} \leq \frac{10h + 4}{9h + 3}.$$

This is at most $\frac{8}{7}$ for $h \geq 2$. For $h = 1$, the planted clique tree inequality reduces to the 3-regular 3-path inequality whose strength is equal to $\frac{7}{6}$ in agreement with Theorem 5.

5 Interpretation

We give three different interpretations of the results of the previous section.

1. Experimental results show that strong cutting-plane algorithms work very well for the travelling salesman problem. This seems to indicate that the intersection of all

the known classes of inequalities for $STSP$ or $GTSP$ approximates it very closely. The results of the previous sections show that if one optimizes any nonnegative cost function over all subtour elimination constraints, all clique tree inequalities, all path inequalities, all crown inequalities, all hypohamiltonian inequalities and all planted clique tree inequalities, the resulting bound is at most $\frac{4}{3}$ times the Held-Karp lower bound. This is some strongly supporting evidence for Conjecture 1.

2. If the clique tree inequalities and the subtour elimination constraints are sufficient to prove optimality for some instance of the TSP, then the Held-Karp lower bound must be within a factor of $\frac{8}{7} < 115\%$ of optimal. This gives some explanation of the tightness of the bound in practice.

3. If one considers a special instance of the travelling salesman problem in which the cost function consists of the left-hand-side of a p-regular 3-path inequality with p large then all the classes of valid inequalities studied in this paper, except the path inequalities themselves, will do badly on this instance. Indeed, the class of comb inequalities for example will close at most

$$\frac{\frac{10}{9}-1}{\frac{4}{3}-1} = \frac{1}{3} = 33.3\%$$

of the duality gap. This is not impressive. Similarly the class of clique trees will close at most

$$\frac{\frac{8}{7}-1}{\frac{4}{3}-1} = \frac{3}{7} < 43\%$$

of the duality gap. In fact, to close more than 43% of the duality gap, one must use the path inequalities, or some class of inequalities which is not considered in this paper.

6 Concluding remarks

The results of the previous sections apply to $GTSP$. One might wonder whether the worst-case improvements remain unchanged when one adds the degree constraints $x(\delta\{i\}) = 2$ for all $i \in V$ and restricts one's attention to cost functions satisfying the triangle inequality. We believe so but have been unable to prove it. The result would follow immediately if one could prove that the degree constraints never affect the value of the relaxation when the cost function satisfies the triangle inequality. In the two extreme cases (the subtour polytope and $STSP$), this is indeed correct (for the subtour polytope, see [11]).

In this paper, we have provided a measure to evaluate relaxations for $GTSP$. It would be worthwhile to investigate other measures. In particular, one could try to compare the volumes of various relaxations for the TSP.

Finally, since the TSP is NP-hard, one should not expect to obtain a simple description of the facets of $GTSP$. Nevertheless, one could hope for the derivation of properties of them which allow to prove an upper bound on the strength of any facet-defining valid inequality. This could lead to a proof of Conjecture 1.

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