# On the determinant of a uniformly distributed complex matrix

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**Abstract**

We derive the joint density for the singular values of a random complex matrix $A$ uniformly distributed on $\|A\|_F = 1$. This joint density allows us to obtain the conditional expectation of $\det(A^H A) = |\det A|^2$ given the smallest singular value. This result has been used by Shub and Smale in their analysis of the complexity of Bezout's Theorem.

## 1 Introduction

Let $A$ be a random $n \times n$ complex matrix uniformly distributed on the sphere $1 = \|A\|_F = \sum_{ij} |A_{ij}|^2$. In order to analyze a special mathematical model regarding the probability that a numerical analysis problem is difficult, Demmel [2] and Edelman [3, 4] have investigated the distribution of the scaled condition number $\kappa_D(A) \equiv \|A\|_F \|A^{-1}\|$. Because of the scale invariance of the condition number and special properties of the normal distribution, it is equivalent to assume that the random matrices are generated with independent elements from a complex standard normal distribution i.e., a distribution with independent real and imaginary parts with the standard normal distribution. This quantity also arises in the multivariate analysis of variance (MANOVA) as described in multivariate analysis books such as [1].

A related quantity is studied by Shub and Smale [7] in their study of ill-conditioned polynomial systems. Their work requires an estimate for the conditional expectation of $|\det A|^2$ given $\sigma_{\min}$ the smallest singular value of $A$. In a previous study [3], we obtained the joint distribution of the singular values of $A$ up to a constant which we were not able to determine exactly. In this paper, we derive the constant, and then calculate the conditional expectation of the absolute squared determinant given the smallest singular value.

For convenience, we state our results in terms of the eigenvalues $\lambda_i$ of the matrix $A^H A$. These eigenvalues are the squares of the singular values $\sigma_i$ of $A$.

Our result is

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**Theorem 1.1**

$$E(\det A^H A \mid \sigma_{\min}^2 = \lambda) = \sum_{r=0}^{n-1} \lambda^{n-r}(1-n\lambda)^{n^2+r-2} \frac{\Gamma(n^2)\Gamma(n+1)\Gamma(n+2)}{\Gamma(r+1)\Gamma(n-r)\Gamma(n^2+r-1)\Gamma(n+2-r)}.$$

## 2 Joint Distribution of the $\lambda_i$

**Lemma 2.1** *The joint distribution of the squares of the singular values of $A$: $\lambda_1 \leq \ldots \leq \lambda_n$ is*

$$\frac{\Gamma(n^2)}{\left(\Pi_{j=1}^n \Gamma(j)\right)^2} \prod_{i<j} (\lambda_i - \lambda_j)^2,$$

*where $\sum \lambda_i = 1$.*

**Proof**

The joint density for a matrix $A$ uniformly distributed on $||A||_F$ is

$$\frac{2}{Vol(S^{2n^2-1})} \delta(\mathrm{tr}(A^H A) - 1) = \frac{\Gamma(n^2)}{\pi^{n^2}} \delta(\mathrm{tr}(A^H A) - 1),$$

where $\delta$ is the Dirac delta function. The constant is chosen so that the integral is one.

The formula for the joint density of the $\lambda_i$ may be obtained most easily by using the Jacobians for the complex QR decomposition, followed by the complex Cholesky decomposition. We omit the details of the computation here, but point out they may be obtained from the prescription on [3, pages 33–36]. The basic techniques are similar to those used to calculate the distribution of the elements and eigenvalues of a Wishart matrix as described in such books as [6].

An alternative derivation of the constant may be based on formula (17.6.5) of Mehta's book [5, p.354] by taking $\alpha = \gamma = 1$:

$$\int_{R^n} \exp(-\Sigma_{i=1}^n x_i) \prod_{i \neq j} (x_i - x_j)^2 dx = \prod_{j=0}^{n-1} \Gamma(j+2)\Gamma(j+1). \tag{1}$$

When we are integrating over the simplex $\Sigma_n = \{x_1 + \cdots + x_n = 0\}$, we write $dx$ to denote $dx_1 \ldots dx_{n-1}$ the last variable, $x_n$ being determined by the others.

$$\begin{aligned}
\int_{R^n} \exp(-\Sigma_{i=1}^n x_i) \Pi_{i \neq j} (x_i - x_j)^2 dx &= \int_{s=0}^{\infty} e^{-s} \int_{\Sigma_n} \Pi(x_i - x_j)^2 dx ds \\
&= \int_{s=0}^{\infty} e^{-s} s^{n(n-1)} s^{n-1} \int_{\Sigma_n} \Pi(x_i - x_j)^2 dx ds \\
&= \int_{s=0}^{\infty} e^{-s} s^{n^2-1} \int_{\Sigma_n} \Pi(x_i - x_j)^2 dx ds \\
&= \Gamma(n^2) \int_{\Sigma_n} \Pi(x_i - x_j)^2 dx
\end{aligned}$$

$\square$

## 3 The Conditional Expectation of $\det(A^H A)$

Let $\lambda = \lambda_1$. Weighing the joint density in Lemma 2.1 by the determinant and distinguishing $\lambda$ gives

$$E(\det A^H A \mid \sigma_{\min}^2 = \lambda) = \frac{\Gamma(n^2)}{\left(\Pi_{j=1}^n \Gamma(j)\right)^2} \lambda \prod_{j>1} (\lambda - \lambda_j)^2 \prod_{1<i<j} (\lambda_i - \lambda_j)^2 \lambda_2 \dots \lambda_n. \tag{2}$$

Let $x_j = \lambda_j - \lambda$ transforming (2) into

$$\frac{\Gamma(n^2)}{\left(\Pi_{j=1}^n \Gamma(j)\right)^2} \lambda \prod_{j>1} x_j^2 \prod_{1<i<j} (x_i - x_j)^2 \prod_2^n (x_j + \lambda),$$

where $\sum_{j=2}^n x_j = 1 - n\lambda$. We divide by $\Gamma(n)$ to remove the ordering of the $x_i$ and make the change of variables $y_j = x_{j+1}/(1 - n\lambda)$ obtaining

$$\frac{\Gamma(n^2)}{\Gamma(n)\left(\Pi_{j=1}^n \Gamma(j)\right)^2} \lambda (1 - n\lambda)^{n^2+n-3} \int_{\Sigma_{n-1}} \Pi y_j^2 \Pi (y_i - y_j)^2 \Pi (y_j + \frac{\lambda}{1 - n\lambda}) dy. \tag{3}$$

We explain the exponent $n^2 + n - 3$ of the term $(1 - n\lambda)$ as follows: $2(n-1)$ from the $x_j^2$, $(n-1)(n-2)$ from the $(x_i - x_j)^2$, $n - 1$ from the $x_j + \lambda$ and $n - 2$ from the volume element on the simplex $\Sigma_{n-1}$. It is the goal in this note to compute an exact expression for (3). Notice that this expression is a polynomial of degree $n^2 + n - 2$ in $\lambda$.

**Lemma 3.1** *Let*

$$I_n^r = \int_{R_+^n} \exp(-\Sigma_{j=1}^n y_j) \Delta^2 (\Pi y_j^2)(y_1 \dots y_r) dy, \tag{4}$$

*where* $\Delta = \Pi_{i<j}(y_i - y_j)$. *We have*

$$I_n^r / I_n^0 = \Gamma(n+3)/\Gamma(n+3-r)$$

*and*

$$I_n^0 = \Pi_{j=0}^{n-1} \Gamma(j+2)\Gamma(j+3) = \Pi_{j=1}^{n+1} \left(\Gamma(j)^2\right) \Gamma(n+2). \tag{5}$$

**Proof** $I_n^0$ is in Mehta, [5, p.354] as (17.6.5) with $\gamma = 1$ and $\alpha = 3$.

Mehta [5, p.340,(17.1.4)] also reports on Aomoto's extension of Selberg's integral:

$$\frac{\int_{[0,1]^n} x_1 x_2 \dots x_r \Delta^2 \Pi x_j^2 (1 - x_j)^{\beta - 1}}{\int_{[0,1]^n} \Delta^2 \Pi x_j^2 (1 - x_j)^{\beta - 1}} = \prod_{j=1}^r \frac{n + j - 3}{2n - j + 2 + \beta}.$$

Using the same idea as in the derivation of (17.6.5) let $x_j = y_j / L$ and $\beta = L + 1$. Taking the limit as $L \to \infty$ we conclude that

$$\frac{I_n^r}{I_n^0} = \Pi_{j=1}^r (3 + n - j) = \Gamma(n+3)/\Gamma(n+3-r).$$

$\square$

**Proof of Theorem 1.1**

Let $I^r_{n,\Delta}$ denote the integrand defining $I^r_n$ in (4) with two modifications: remove the exponential term and integrate over the simplex $\Sigma_n = \Sigma_{j=1}^n y_j = 1$. Here the $y_j$ are unordered.

Make the change of variables

$$s = \Sigma y_i,\ t_i = y_i/s,\ y_i = t_i s.$$

Then,

$$I^r_n = I^r_{n,\Delta} \int_{s=0}^{\infty} e^{-s} s^{n^2+2n+r-1} ds$$

so that

$$\begin{aligned} I^r_n &= I^r_{n,\Delta}\Gamma(n^2+2n+r), \\ I^r_{n,\Delta} &= \frac{I^r_n}{\Gamma(n^2+2n+r)}, \\ &= \frac{I^0_n\Gamma(n+3)}{\Gamma(n^2+2n+r)\Gamma(n+3-r)}. \end{aligned}$$

Let

$$\begin{aligned} f(\theta) &= \int_{\sigma_{n-1}} \Pi y_j^2 \Delta^2 \Pi_j (y_j+\theta) dy \\ &= \sum_{r=0}^{n-1} \theta^{n-r-1} \binom{n-1}{r} I^r_{n-1,\Delta} \\ &= \left( \sum_{r=0}^{n-1} \theta^{n-r-1} \binom{n-1}{r} \frac{\Gamma(n+2)}{\Gamma(n^2+r-1)\Gamma(n+2-r)} \right) I^0_{n-1}. \end{aligned}$$

We therefore conclude from (3) that $E(\det A^H A \mid \sigma^2_{\min} = \lambda) = f(\lambda/(1-n\lambda))$. Plugging in the value from (5) we obtain the desired result

$$E(\det A^H A \mid \sigma^2_{\min} = \lambda) = \sum_{r=0}^{n-1} \lambda^{n-r}(1-n\lambda)^{n^2+r-2} \frac{\Gamma(n^2)\Gamma(n+1)\Gamma(n+2)}{\Gamma(r+1)\Gamma(n-r)\Gamma(n^2+r-1)\Gamma(n+2-r)}.$$

$\square$

Integrating this from $\lambda = 0$ to $1/n$ gives the expected determinant squared of a uniformly distributed complex matrix:

$$E_n \equiv E(\det A^H A) = \frac{\Gamma(n^2)\Gamma(n+1)\Gamma(n+2)}{\Gamma(n^2+n)} \sum_{r=0}^{n-1} \frac{\Gamma(n-r+1)}{\Gamma(r+1)\Gamma(n-r)\Gamma(n+2-r)n^{n-r+1}} = \binom{n}{n^2+n-1}^{-1}$$

The first ten values of $1/E_n$ are $\{1, 10, 165, 3876, 118755, 4496388, 202927725, 10639125640, 635627275767, 42634215112710\}$.

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