A Note on Optical Routing on Trees

S. Ravi Kumar [a,1], Rina Panigrahy [b,2], Alexander Russell [c,3], Ravi Sundaram [d,4]

[a] *Dept. of Computer Science, Cornell University, Ithaca, NY 14853.*

[b] *Lab. for Computer Science, M.I.T., Cambridge, MA 02139.*

[c] *Dept. of Mathematics, M.I.T., Cambridge, MA 02139.*

[d] *Lab. for Computer Science, M.I.T., Cambridge, MA 02139.*

Abstract

Bandwidth is a very valuable resource in wavelength division multiplexed optical networks. The problem of finding an optimal assignment of wavelengths to requests is of fundamental importance in bandwidth utilization. We present a polynomial-time algorithm for this problem on fixed constant-size topologies. We combine this algorithm with ideas from Raghavan and Upfal [15] to obtain an optimal assignment of wavelengths on constant degree undirected trees. Mihail, Kaklamanis, and Rao [14] posed the following open question: what is the complexity of this problem on directed trees? We show that it is NP-complete both on binary and constant depth directed trees.

Keywords: Algorithms, Combinatorial Problems, Computational Complexity, Interconnection Networks.

1 Introduction

Motivation. Developments in fiber-optic networking technology using *Wavelength Division Multiplexing* (WDM) have finally reached the point where it

[1] Supported by ONR Young Investigator Award N00014-93-1-0590. This work was done while the author was visiting M.I.T. Email: `ravi@cs.cornell.edu`.

[2] Supported by WWW Consortium at MIT. Email: `rinap@theory.lcs.mit.edu`.

[3] Supported by an NSF Graduate Fellowship and grants NSF 92-12184, AFOSR F49620-92-J-0125, and DARPA N00014-92-1799. Email: `acr@theory.lcs.mit.edu`.

[4] Supported by grants NSF 92-12184, AFOSR F49620-92-J-0125, and DARPA N00014-92-1799. Email: `koods@theory.lcs.mit.edu`.

is being considered as the most promising candidate for the next generation of wide-area backbone networks. These are highly flexible networks capable of supporting tens of thousands of users and providing capacities on the order of gigabits-per-second per user [4,7,16]. WDM optical networks utilize the large bandwidth available in optical fibers by partitioning it into several channels each at a different optical wavelength [2,4,10,11].

The typical optical network consists of routing nodes interconnected by point-to-point fiber-optic links. Each link supports a certain number of wavelengths. A *lightpath* is an optical path between two nodes on a specific wavelength. *Photonic switching*, also known as *dynamic wavelength routing*, is the setting up of lightpaths. The routing nodes are capable of photonic switching [3,5,17]. Though the total number of wavelengths is limited, nevertheless, it is possible to build a transparent wide-area optical network by spatial reuse of wavelengths (see Figure 1).

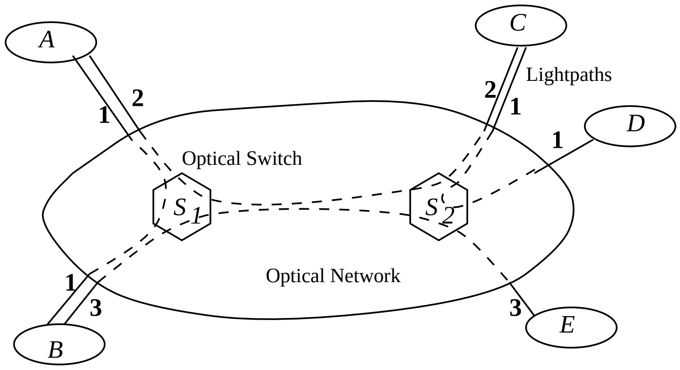

Fig. 1. Example of an All-Optical Wavelength Routing Network

The Problem. The fundamental problem in bandwidth utilization in optical networks can be modeled as follows:

> Optical Routing: Given a (directed/undirected) graph G and a set P of n requests (each of which is a pair of nodes to be connected by a directed/undirected path) find a set of paths corresponding to these requests and an assignment of wavelengths to the requests so as to minimize the total number of wavelengths used. The assignment must ensure that different paths allotted the same wavelength must be edge-disjoint.

In practice, the topology of the optical network stays fixed over time and the size of the set of requests to be routed is a varying parameter. Hence, it is interesting to study Optical Routing on fixed-size arbitrary topologies.

Previous Work. Optical Routing has been studied in great detail on a number of different and fundamental topologies by Raghavan and Upfal [15].

This problem is NP-complete for undirected trees by reduction from edge coloring. They gave a 9/8-approximation algorithms for undirected trees and a 2-approximation algorithm for rings. They also present approximation algorithms for expanders, meshes, and bounded-degree graphs. Subsequently, Mihail, Kaklamanis, and Rao [14] consider the problem on directed trees (these are trees obtained by replacing each edge of an undirected tree by two directed edges in opposite directions). They obtain a 15/8-approximation algorithm using potential function arguments. Aumann and Rabani [1] consider routing permutations on arrays, hypercubes, and arbitrary bounded degree networks, while Bermond et. al. [12] study the problems of broadcasting and gossiping in optical networks. Goldberg, Jerrum, and Mackenzie prove a lowerbound for routing h-relations in complete optical networks [8].

Our Results. We present an algorithm for OPTICAL ROUTING on fixed constant-size topologies which has running time polynomial in the number of requests (Section 2). We combine this algorithm with ideas from Raghavan and Upfal [15] to obtain an optimal assignment of wavelengths on constant degree undirected trees (Section 4). Mihail, Kaklamanis, and Rao [14] posed the following open question: what is the complexity of this problem on directed trees? We show that it is NP-complete both on constant depth (Section 3) and binary directed trees (Section 4).

Notation. For a graph (directed/undirected) $G = \langle V; E \rangle$ on k vertices, let $\chi'(G)$ denote the *chromatic index* (i.e., the edge color number). In any edge coloring χ', let $\chi'(e)$ denote the color of any $e \in E$. For $u, v \in V$, denote an undirected path between u and v by $u \sim v$ and denote a directed path from u to v by $u \rightsquigarrow v$. When G is a tree, for any given $u, v \in V$, the path $u \sim v$ is unique and hence OPTICAL ROUTING is equivalent to path coloring on trees. A directed is thought to have edge pairs of the form $u \rightsquigarrow v$ and $v \rightsquigarrow u$. Given an instance of OPTICAL ROUTING with a graph G and a (multi)set $P(|P| = n)$ of paths, let $\Lambda(G, P)$ denote the number of wavelengths used. In such a wavelength assignment, let $\lambda(p)$ denote the wavelength of a path p. Let $\Lambda^*(G, P)$ denote the optimal number of wavelengths for the set of requests P. Note that wavelengths can also be interpreted as colors.

2 Constant Size Graphs

We now present an exact polynomial-time algorithm for the problem of OPTICAL ROUTING on a fixed constant-size topology, G (i.e., k is constant). This problem is non-trivial and a combinatorial algorithmic approach seems difficult. G may be directed or undirected. We present the algorithm only for undirected graphs. The algorithm for directed graphs is a straightforward modification.

Theorem 1 Optical Routing *on constant-size topologies is in* P.

Proof. Let r_{ij} denote the number of paths from node i to j. Note that r_{ij} could potentially be as large as n. Let a path-matching be a collection of edge-disjoint simple paths. Let $\mathcal{M}_G = \{M \mid M \text{ is a path-matching in } G\}$ be the set of path-matchings in G. Note that since G is constant-size, so is $\mathcal{M}_G$.

Consider the following integer program (IP):

$$\min \sum_{M \in \mathcal{M}_G} x_M \qquad \text{subject to} \qquad \forall i, j, \quad \sum_{i \sim j \in M} x_M \geq r_{ij}.$$

It is easy to see that the above IP models the problem of Optical Routing exactly. A solution to the IP yields $\Lambda^*(G, P)$, the optimum number of wavelengths needed. Though the IP has terms in the constraints which are linear in n, the optimizing function has only a constant number of variables. This means we can solve the IP exactly using Lenstra's polynomial-time algorithm for integer programs in fixed dimension [13]. □

3 Constant Depth Trees

For undirected trees of constant depth, the problem of coloring paths is known to be NP-complete. For directed trees, however, no hardness result was known before. We now give a reduction that shows that coloring paths on directed trees of depth 3 is NP-complete. To make it easy, first, we give the reduction for undirected trees and then we show how to modify the construction for directed trees.

Theorem 2 *Coloring paths on undirected trees of depth 2 is* NP*-complete.*

Proof. We reduce edge coloring to this problem: given an undirected graph $G = \langle V; E \rangle$ and an integer k, is $\chi'(G) = k$? We build an undirected tree T as follows: $T = \langle V \cup \{r\}; \{(r, v) \mid v \in V\} \rangle$. Note that the degree of r in T is $|V|$ and depth of T is 2. The set of paths is $P = \{v_i \sim v_j \rangle \mid (v_i, v_j) \in E \wedge i < j\}$. In other words, the set of paths in the tree is just the edges in the graph. Clearly, all the paths have to go via r in T.

If $\chi'(G) = k$, then using such a coloring, paths in P on T can be colored using k colors: simply color the path $p = v_i \sim v_j$ as $\lambda(p) = \chi'(v_i, v_j)$. The fact that

λ is a valid path coloring follows from the fact that χ' is a valid edge coloring and clearly $\Lambda(T, P) = k$. Conversely, if paths in P on T are colored using k colors, then, for $i < j$, set $\chi'(v_i, v_j) = \lambda(v_i \sim v_j)$. Thus, all edges leaving any vertex in G are colored differently. $\square$

To extend the above result to directed trees, we need the following gadget $T_{r,x,y}$: a depth 2 directed tree with a root r and children x, y and with edge set $\{r \rightsquigarrow x, x \rightsquigarrow r, r \rightsquigarrow y, y \rightsquigarrow r\}$. Let $P_{r,x,y,k}$ denote the multiset $\{x \rightsquigarrow y\}$ of multiplicity k. All paths in $P_{r,x,y,k}$ have to be colored with k different colors, i.e., $\Lambda^*(T_{r,x,y}, P_{r,x,y,k}) = k$.

Theorem 3 *Coloring paths on directed trees of depth 3 is* NP*-complete.*

Proof. We reduce edge coloring to this problem. We use a stronger version of edge coloring where we assume the graph is k-regular (in fact, even for $k = 3$, i.e., cubic graphs, the problem is NP-complete [9]). Given a k-regular graph G, is $\chi'(G) = k$? First, we construct the tree T essentially as in Theorem 2, but with undirected edges replaced by directed edges in both directions. Now, set $P = \{v_i \rightsquigarrow v_j, v_j \rightsquigarrow v_i \mid (v_i, v_j) \in E\}$.

T and P are almost good for us. It is easily seen that if $\chi'(G) = k$, then the paths in P can be colored with k colors by assigning $\lambda(v_i \rightsquigarrow v_j) = \lambda(v_j \rightsquigarrow v_i) = \chi'(v_i, v_j)$ (since these directed paths don't interfere with each other, this is valid). The converse would be easy if for all path pairs $\lambda(v_i \rightsquigarrow v_j) = \lambda(v_j \rightsquigarrow v_i)$ in which case setting $\chi'(v_i, v_j) = \lambda(v_i \rightsquigarrow v_j)$ would suffice. Unfortunately, this is not true and it could happen that $\lambda(v_i \rightsquigarrow v_j) \neq \lambda(v_j \rightsquigarrow v_i)$. To circumvent this problem, we use the gadget described above to construct a new tree T' and new set of paths P'.

For path pair $v_i \rightsquigarrow v_j, v_j \rightsquigarrow v_i$ in P, tack the gadgets $T_{v_i,v_{ij1},v_{ij2}}$ and $T_{v_j,v_{ji1},v_{ji2}}$ to obtain T'. Note that the depth of T' is 3. Now, construct the multiset P' from P by replacing $v_i \rightsquigarrow v_j$ and $v_j \rightsquigarrow v_i$ by $v_{ij1} \rightsquigarrow v_{ji2}$ and $v_{ji1} \rightsquigarrow v_{ij2}$ and adding $P_{v_i,v_{ij1},v_{ij2},k-1}$ and $P_{v_j,v_{ji1},v_{ji2},k-1}$. This modified T' ensures that in a k-coloring of P', $\lambda(v_{ij1} \rightsquigarrow v_{ji2}) = \lambda(v_{ji1} \rightsquigarrow v_{ij2})$ since $\Lambda^*(P_{v_i,v_{ij1},v_{ij2},k-1}) = k - 1$. $\square$

Figure 2 illustrates the reduction for a 3-node graph. It is interesting to note that coloring paths on directed trees of depth 2 is in P (in contrast to undirected trees). This follows from an obvious reduction to edge coloring bipartite graphs [14].

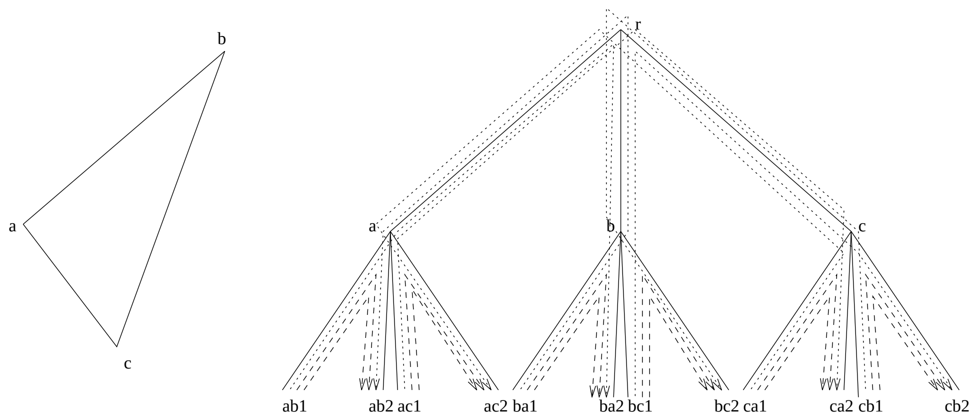

Fig. 2. Reduction to a Constant Depth Directed Tree

4 Constant Degree Trees

First, we show that for undirected trees of constant degree, the problem of coloring paths is in P. Then, we show that for directed trees, this problem is NP-complete even for binary trees.

Theorem 4 *Coloring paths on undirected bounded-degree trees is in* P*.*

Proof. Let $T = \langle V; E\rangle$. The problem now is to assign colors in an optimal fashion to paths in P. Now observe that the following nice decomposition result holds: let the removal of edge $e \in E$ result in two trees $T_1 = \langle V_1; E_1\rangle$ and $T_2 = \langle V_2; E_2\rangle$. P is partitioned into P_1 (the paths entirely in T_1), P_2 (the paths entirely in T_2), and P_{12} (the paths that go through e). Consider the trees $\langle V_1; E_1 \cup \{e\}\rangle$ with paths $P_1 \cup P_{12}$ and $\langle V_2; E_2 \cup \{e\}\rangle$ with paths $P_2 \cup P_{12}$. If we can color both the above instances optimally then we can combine the colors of the two to get an optimal coloring of P on T. Thus we need only to solve the problem on stars (trees with one central vertex and many leaves) and then we can put the solutions together to get a solution for the whole. But since T is constant degree, the stars we get from breaking up the tree are constant-sized and so we can use the results from Theorem 1 to get optimal solutions for these stars.

Thus we have an exact polynomial-time algorithm for the problem of Optical Routing on a constant-degree undirected trees. $\square$

To prove the next theorem, we need some basic results from vertex coloring interval and circular-arc graphs. Recall that an interval (resp. a circular-arc) graph is one whose vertices can be represented as arcs on a line (resp. circle) such that two arcs intersect if and only if those vertices have an edge. Vertex coloring intervals graphs can be done in linear time. Vertex coloring circular-arc graphs is NP-hard [6].

To obtain the result on directed binary trees, we note the following simple transformation β_v on an instance T and P: Let v be a leaf node in a directed tree T. Let $v_i \rightsquigarrow v, v \rightsquigarrow v_i \in P$ for $i = 1, \ldots, p$. Then, $\beta_v(T)$ is T with a complete binary tree of $\lg p$ levels fixed to the node v. Let the leaves of this complete binary tree be labeled $u_1, \ldots, u_p$. Then, $\beta_v(P)$ is P with each $v_i \rightsquigarrow v, v \rightsquigarrow v_i$ pair replaced by $v_i \rightsquigarrow u_i, u_i \rightsquigarrow v_i$ for $i = 1, \ldots, p$. It can be seen that $\Lambda^*(T, P) = \Lambda^*(T', P')$.

Theorem 5 *Coloring paths on directed binary trees is* NP*-complete.*

Proof. We reduce vertex coloring of circular-arc graphs to this problem. Given a circular-arc graph G and an integer k, is $\chi(G) = k$? Our idea will be to embed a circle on the tree. In the circular-arc representation of G, first we "cut" G at any two points p_1, p_2 on the circle to partition G into two "pieces" G_1, G_2. Let G_{12} (resp. G_{21}) be set of arcs that were "cut" at p_1 (resp. p_2) and hence occur in both G_1 and G_2. Let $|G_{12}| = k_1, |G_{12}| = k_2$. Note that G_1 and G_2 are interval graphs. It is easy to embed an interval graph as paths on a binary tree – the arcs simply correspond to paths on a degenerate tree. A coloring of paths directly corresponds to coloring of an interval graph. So, it is easy to consider a long enough (degenerate) directed tree T and construct P in the following manner. P consists of arcs in G_1 interpreted in the "upward" direction and of arcs in G_2 interpreted in the "downward" direction on T. Since T is degenerate, let it be rooted and let the leaves be, say a and b. If $x \rightsquigarrow p_1 \rightsquigarrow y$ was an arc in G that was cut, we have $x \rightsquigarrow a, a \rightsquigarrow y \in P$, and we have to ensure that $\lambda(x \rightsquigarrow a) = \lambda(a \rightsquigarrow y)$. To do this, we first applying the transformation discussed above to obtain $T' = \beta_b(\beta_a(T))$ and $P' = \beta_b(\beta_a(P))$. Note that this transformation preserves the coloring but makes the tree binary. Let the leaves of this binary tree T' be $a_1, \ldots, a_{k_1}$ and $b_1, \ldots, b_{k_2}$. As in the proof of Theorem 3, we tack the gadget $T_{a_i, a_{i1}, a_{i2}}$ and $T_{b_j, b_{j1}, b_{j2}}$ to T' for each $i = 1, \ldots, k_1, j = 1, \ldots, k_2$ to obtain T''. P'' is obtained from P' by replacing every $z \rightsquigarrow a_i$, $z \rightsquigarrow b_j$ by by $z \rightsquigarrow a_{i2}$, $z \rightsquigarrow b_{j2}$ and adding $P_{a_i, a_{i1}, a_{i2}, k-1}$, $P_{b_j, b_{j1}, b_{j2}, k-1}$ for $i = 1, \ldots, k_1 - 1, j = 1, \ldots, k_2 - 1$.

Now, if $\chi(G) = k$, it is easy to see $\Lambda(T'', P'') = k$. Conversely, if $\Lambda(T'', P'') = k$, then, the tacked gadgets ensure that the arcs that were cut are colored with same color. $\square$

Figure 3 illustrates the reduction for a 3-node circular-arc graph.

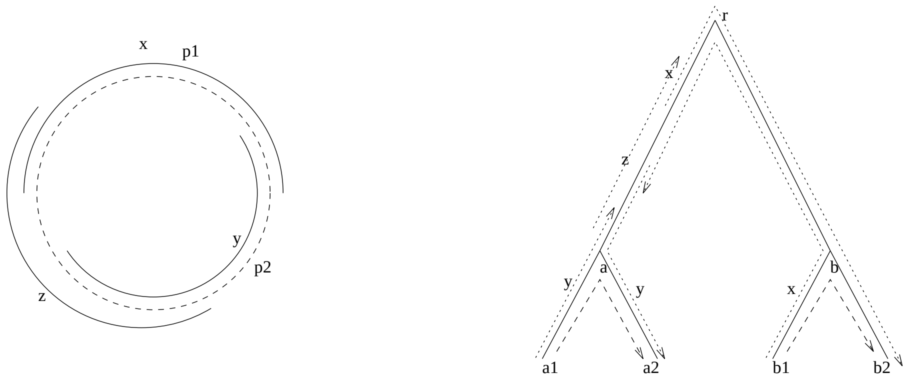

Fig. 3. Reduction to a Binary Directed Tree

5 Some Remarks

(i) An interesting parameter is the thickness denoted $L_{\max}$. Given a tree T and set of paths P, $L_{\max}$ is defined to be the maximum thickness of the paths at any point in T. When $L_{\max} = 1$, one color suffices since there is no overlap of paths. When $L_{\max} = 2$, we can show that the set of paths can be decomposed into a tree of cycles. Hence, either 2 or 3 colors suffice and the problem is in P. Theorem 3 which uses reduction from edge-coloring of cubic graphs shows that the problem is NP-complete when $L_{\max} = 3$.

(ii) Suppose the tree is oriented, i.e., embedded rigidly on the plane. We can show that even if at all internal nodes, all the paths are from the left child to right child, the problem of Optical Routing on directed trees is still NP-complete.

(iii) The problem of obtaining a better than $7L_{\max}/4$ algorithm for binary trees or better than $15L_{\max}/8$ for general trees [14] is still tantalizingly open.

Acknowledgement

The authors wish to thank Prof. David Shmoys (Cornell) for his solution to the problem of optical routing on fixed constant-size topologies.

References

[1] Y. Aumann and Y. Rabani. Improved bounds for all optical routing. *Proc. 6th ACM-SIAM Symposium on Discrete Algorithms*, pages 567–576. SIAM, 1995.

[2] R. A. Barry and P. A. Humblet. Bounds on the number of wavelengths needed in WDM networks. *LEOS '92 Summer Topical Meeting Digest*, pages 21–22,

1992.

[3] C. Chen and S. Banerjee. Optical switch configuration and lightpath assignment in wavelength routing multihop lightwave networks. *Proc. IEEE Infocom*, pages 1300–1307, 1995.

[4] N. K. Cheung, K. Nosu, and G. Winzer, editors. *IEEE JSAC: Special Issue on Dense WDM Networks*, volume 8, 1990.

[5] I. Chlamtac, A. Ganz, and G. Karmi. Lightpath communications: An approach to high bandwidth optical WANs. *IEEE Transactions on Communication*, 40(7):1171–1182, 1992.

[6] M. R. Garey, D. S. Johnson, G. L. Miller, and C. H. Papadimitriou. The complexity of coloring circular arcs and chords. *SIAM Journal of Algebraic and Discrete Methods*, 1(2):216–227, 1980.

[7] P. E. Green. *Fiber-Optic Networks*. Prentice-Hall, 1992.

[8] L. A. Goldberg, M. Jerrum, and P. D. MacKenzie. A lower bound for routing on a completely connected optical communication parallel computer. *SIAM Journal on Computing*, to appear.

[9] I. Holyer. The NP-completeness of edge coloring. *SIAM Journal on Computing*, 10(4):718–720, 1981.

[10] IEEE/OSA. *IEEE/OSA Journal of Lightwave Technology, special issue on Broad-Band Optical Networks*, volume 11, 1993.

[11] M. Irshid and M. Kavehrad. A WDM cross-connected star topology for multihop lightwave networks. *IEEE/OSA Journal of Lightwave Technology*, pages 828–835, June 1992.

[12] J.-C.Bermond, L. Gargano, S. Perennes, A. A. Rescigno, and U. Vaccaro. Efficient collective communication in optical networks. *Proc. 23rd International Colloquium on Automata, Languages, and Programming*, pages 574–585, 1996.

[13] H. W. Lenstra, Jr. Integer programming with a fixed number of variables. *Mathematics of Operations Research*, 8:538–548, 1983.

[14] M. Mihail, C. Kalamanis, and S. Rao. Efficient access to optical bandwidth. *Proc. 36th Annual IEEE Foundations of Computer Science*, pages 134–143, 1995.

[15] P. Raghavan and E. Upfal. Efficient routing in all-optical networks. *Proc. 26th Annual ACM Symposium on the Theory of Computing*, pages 548–557, 1994.

[16] R. Ramaswami. Multi-wavelength lightwave networks for computer communication. *IEEE Communications Magazine*, 31(2):78–88, 1993.

[17] Z. Zhang and A. Acampora. A heuristic wavelength assignment algorithm for multihop WDM networks with wavelength routing and wavelength reuse. *Proc. IEEE Infocom*, pages 534–543, 1994.