6.045J/18.400J: Automata, Computability and Complexity | Prof. Nancy Lynch

Homework 1

Due: February 12, 2007, 5PM | *Elena Grigorescu*

Note: Recitation meets on Thursday, Feb 8 at 10am in 34-302, 1pm in 26-328 and 4pm in 34-304.

Problem 1: Construct truth tables for each of the following formulas. Also, for each pair of formulas, state which of the following holds:

- They are equivalent,
- They are not equivalent, but one implies the other (make sure to state which is which), or
- Neither of the above:

(a) $(p \wedge q) \Rightarrow p$

(b) $p \Rightarrow (q \Rightarrow p)$

(c) $(p \Rightarrow q) \Rightarrow p$

(d) $(p \oplus q) \Rightarrow (\neg p \Leftrightarrow \neg q)$

Problem 2: Fix any finite set S and let the power set of S be denoted by $P(S)$. Let R be the relation between elements of $P(S)$ such that $A\ R\ B$ if and only if there is a bijection between A and B.

(a) Show that R is an equivalence relation.

(b) Define a relation $R_1 \subseteq R$ that is reflexive and symmetric but not transitive.

(c) Define a relation $R_2 \subseteq R$ that is reflexive and transitive but not symmetric.

(d) Define a relation $R_3 \subseteq R$ that is symmetric and transitive but not reflexive.

Problem 3: Proof practice

Part (a). Sipser, Problem 0.10:
Find the error in the following proof that $2 = 1$.
Consider the equation $a = b$. Multiply both sides by a to obtain $a^2 = ab$. Subtract b^2 from both sides to get $a^2 - b^2 = ab - b^2$. Now factor each side, $(a + b)(a - b) = b(a - b)$, and divide each side by $a - b$, to get $a + b = b$. Finally, let $a = 1$ and $b = 1$, which shows that $2 = 1$.

Part (b). Prove that there exists a natural number n_0 such that, for every natural number $n \geq n_0$, there exist natural numbers a, b such that $n = 3a + 7b$.

Part (c). Let function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ be defined recursively as follows:

$$\forall n, k, 0 \leq k \leq n,\ f(n,k) = \begin{cases} 1, & k = 0 \\ 1, & k = n \\ f(n-1,k) + f(n-1,k-1), & \text{for } n \geq 2, 1 \leq k \leq n-1. \end{cases}$$

Prove that for every $n, k, 0 \leq k \leq n,\ f(n,k)$ is equal to the binomial coefficient ("n choose k"), which is defined as

$$\frac{n!}{k!(n-k)!}$$