Dual Problems in Property Testing

Roei Tell, Weizmann Institute of Science

ITCS, January 2016

מכון ויצמן למדע
WEIZMANN INSTITUTE OF SCIENCE

Property Testing

Distinguish between objects that:

- **Have the property**
- **Far from having the property**

A Broad Question

What happens when the property that we want to test is "being far from a set"?

Example:

- Test the property of graphs that are far from being connected

A Broad Question

Distinguish between objects that are:

- **Far from the set**
- **Far from any object that is far from the set**

Distinguish between:

- Graph is far from connected
- Graph is far from any graph that is far from connected

Dual Problems

Standard Problem: $x \in \Pi$ vs $x \in F_\varepsilon(\Pi)$

Dual Problem: $x \in F_\varepsilon(\Pi)$ vs $x \in F_\varepsilon(F_\varepsilon(\Pi))$

- $F_\varepsilon(\Pi)$={ objects that are ε-far from Π }

Dual Problems

Standard Problem: $x \in \Pi$ vs $x \in F_\varepsilon(\Pi)$

$\not\asymp$

Dual Problem: $x \in F_\varepsilon(\Pi)$ vs $x \in F_\varepsilon(F_\varepsilon(\Pi))$

- $F_\varepsilon(\Pi)$={ objects that are ε-far from Π }

Dual Problems: Overview

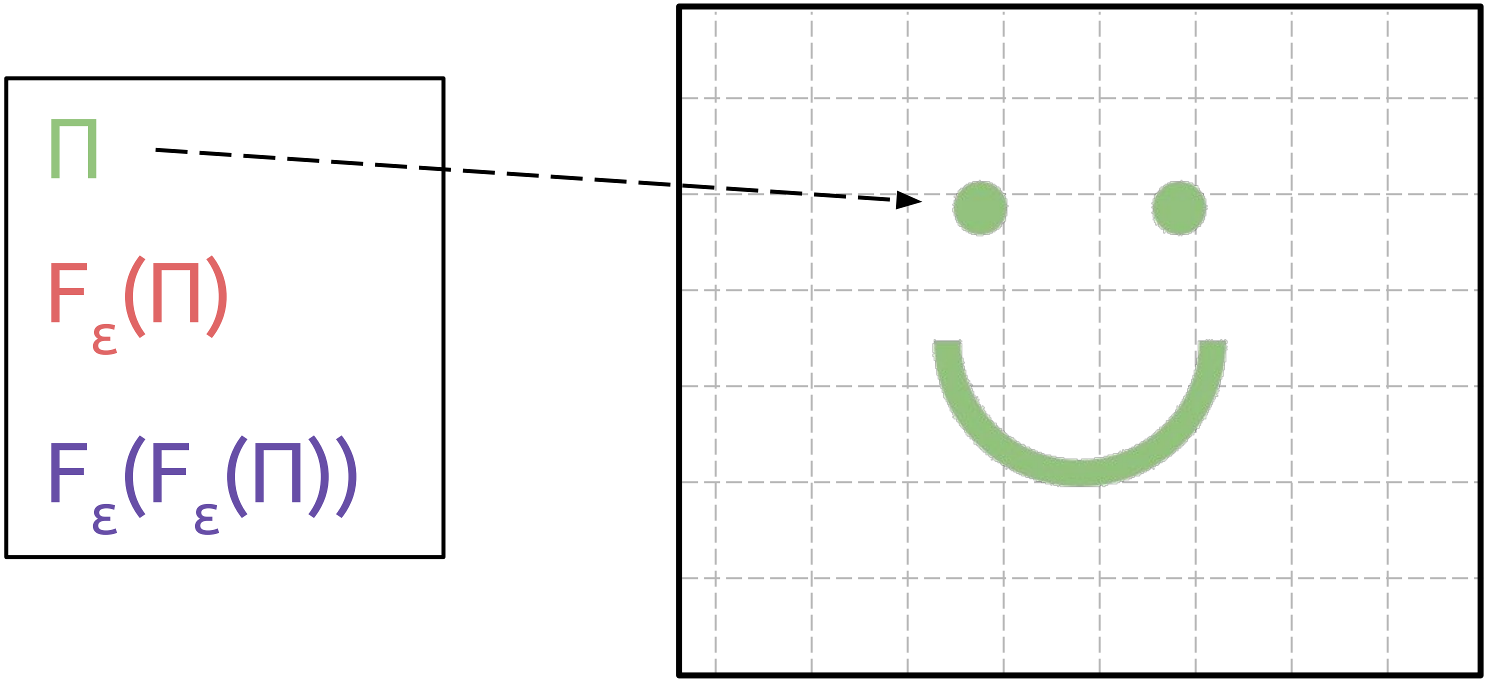
› **Question has not been asked so far**

› **Current work - first exploration:**

- Non-triviality, different from original problems
- Testers for several prominent dual problems
- Identify specific setting of interest - graphs

Non-Triviality of Dual Problems

Non-Triviality: Example

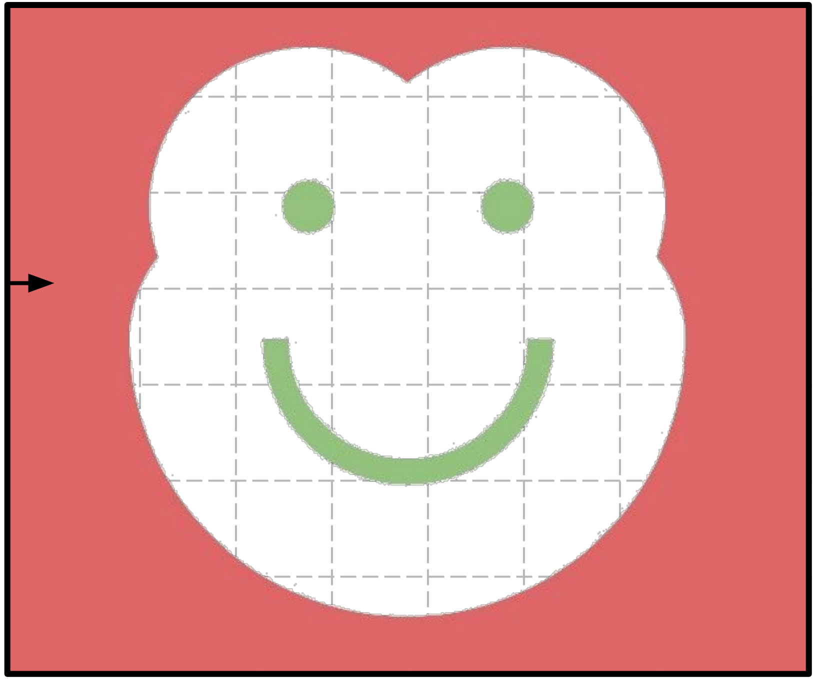


Non-Triviality: Example

Π

$F_\varepsilon(\Pi)$

$F_\varepsilon(F_\varepsilon(\Pi))$

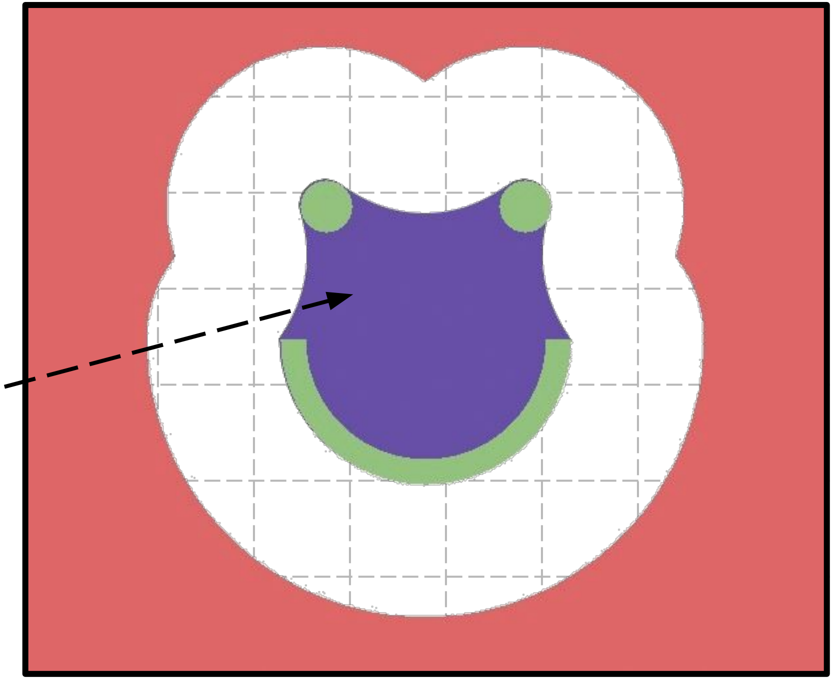


Non-Triviality: Example

Π

$F_\varepsilon(\Pi)$

$F_\varepsilon(F_\varepsilon(\Pi))$



Non-Triviality: Basic Facts

1. **A random* property Π satisfies $F_\varepsilon(F_\varepsilon(\Pi)) \neq \Pi$.**

2. **$\Pi \subseteq F_\varepsilon(F_\varepsilon(\Pi))$, but $F_\varepsilon(F_\varepsilon(\Pi))$ can be much larger than Π.****

3. **$F_\varepsilon(F_\varepsilon(\Pi))$ can contain points that are almost ε-far from Π.**

---

* In $\{0,1\}^n$ and in other classes of metric spaces.

** In $\{0,1\}^n$ the set $F_\varepsilon(F_\varepsilon(\Pi))$ can be $\exp(n)$ larger, even for a small ε.

Non-Triviality: More Examples

$\Pi \neq F_\varepsilon(F_\varepsilon(\Pi))$ › graph properties

- *k*-colorable
- graphs with large clique
- graphs isomorphic to a given graph
- connected
- cycle-free
- bipartite
- ...

(k-colorable, graphs with large clique, graphs isomorphic to a given graph) } dense graphs model

(connected, cycle-free, bipartite) } bounded-degree graphs model

Dual Problems: What we Know

Our Main Results

- **The query complexity of dual testing problems**
  - General lower bounds
  - Testers for specific problems

- **The behavior of “far-from-far” sets**
  - “Far-from-far” closure operator
  - Not presented in this talk

Our Main Results: General Lower Bounds

Thm 1: **The query complexity of any dual problem is lower bounded by that of the original problem.**

Thm 2: **Testing any dual problem with one-sided error requires a linear number of queries (unless $F_\varepsilon(\Pi)=\emptyset$).**

Our Main Results: General Lower Bounds

Thm 1: **The query complexity of any dual problem is lower bounded by that of the original problem.**

Pf:			
	Standard:	Π	$F_\varepsilon(\Pi)$
			$\subseteq$
	Dual:	$F_\varepsilon(\Pi)$	$F_\varepsilon(F_\varepsilon(\Pi))$

Thm 2: **Testing any dual problem with one-sided error requires a linear number of queries (unless $F_\varepsilon(\Pi)=\emptyset$).**

Our Main Results: Specific Upper Bounds

- Testers via equivalence to the original problem ($\Pi = F_\varepsilon(F_\varepsilon(\Pi))$)

Thm 3: The following dual problems are equivalent to the original problems:

1. **Testing whether a string is far from a code.** *
2. **Testing whether a function is far from monotone.** **
3. **Testing whether a distribution is far from uniform.** ***

---

* A code with constant relative distance.

** Functions D→R such that the width of D is bounded (includes functions $\{0,1\}^n \to \{0,1\}$).

*** Generalizes to testing whether a distribution is far from D, if D is from a large class.

Our Main Results: Specific Upper Bounds

› Testers via reductions to tolerant testing

Thm 4: For every ε, it is possible to test whether a graph is:

1. **Far from k-colorable, with Tower(1/ε) queries.** *
2. **Far from being connected, with poly(1/ε) queries.** **
3. **Far from being cycle-free, with poly(1/ε) queries.** **

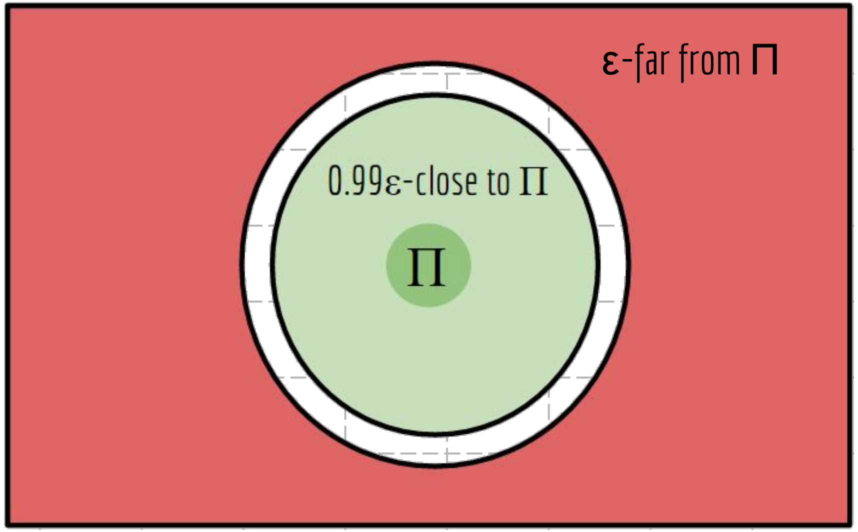
---

* Dense graphs model.

** Bounded-degree graphs model.

Reductions to Tolerant Testing

- **<u>Tolerant testing [PRR]:</u> Distinguish between objects that are**
  - **0.99ε-close to Π**
  - **ε-far from Π**



Reductions to Tolerant Testing

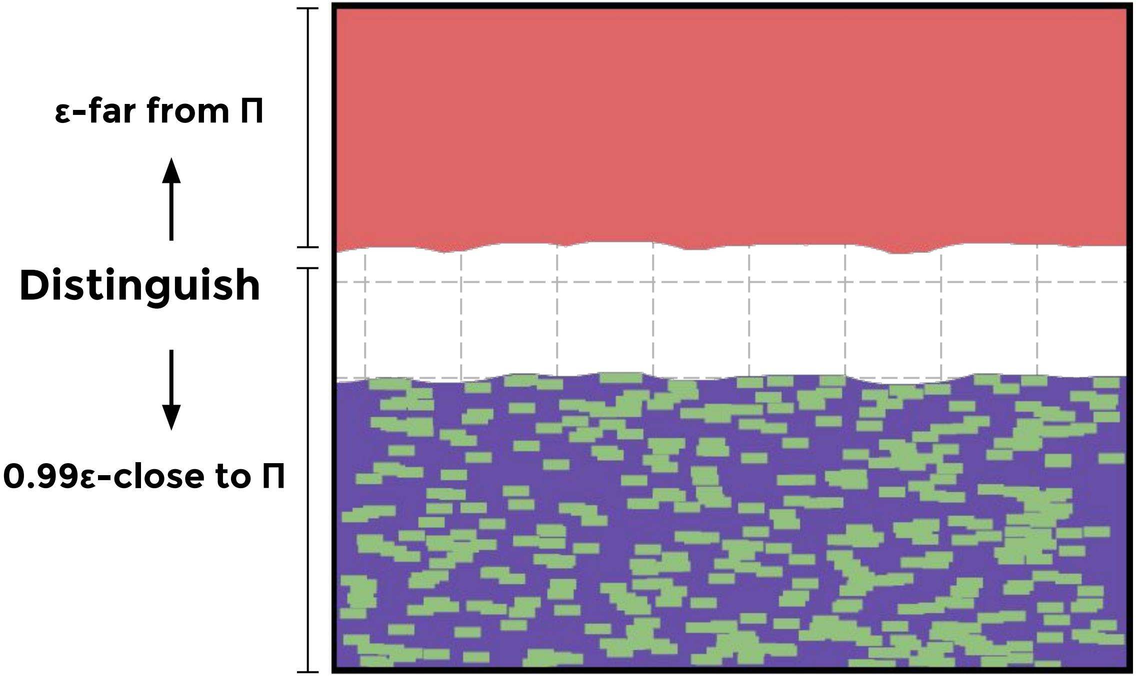
- Tolerant testing [PRR]: Distinguish between objects that are
  - 0.99ε-close to Π
  - ε-far from Π
- Dual reduces to tolerant testing if all points in $F_ε(F_ε(Π))$ are 0.99ε-close to Π

Sometimes $F_\varepsilon(F_\varepsilon(\Pi))$ is 0.99ε-close to Π ...

Π

$F_\varepsilon(\Pi)$

$F_\varepsilon(F_\varepsilon(\Pi))$

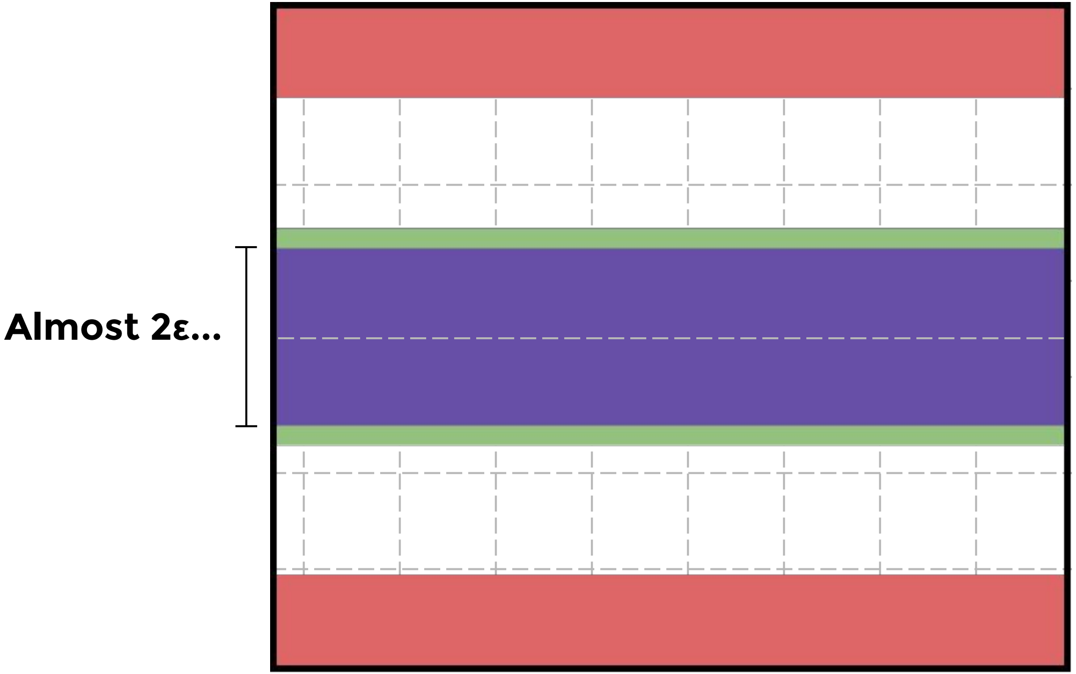


… but $F_\varepsilon(F_\varepsilon(\Pi))$ not always 0.99ε-close to Π

Π

$F_\varepsilon(\Pi)$

$F_\varepsilon(F_\varepsilon(\Pi))$



Generalized Version: ε'-far from ε-far

Standard Problem: $x \in \Pi$ vs $x \in F_\varepsilon(\Pi)$

Generalized Dual Problem: $x \in F_\varepsilon(\Pi)$ vs $x \in F_{\varepsilon'}(F_\varepsilon(\Pi)) \ \forall \varepsilon'$

Generalization

- $F_\varepsilon(\Pi)$={ objects that are ε-far from Π }

Dual Problems: Digest and Current Frontiers

Dual Problems: Key Takeaways

- **Class of natural and unexplored problems**
  - Current work: General lower bounds, six specific testers
- **Different from original problems**
  - And don’t reduce (in general) to tolerant testing
- **Not expecting one global answer**
  - Different settings, different behaviors (graphs vs codes)

Dual Problems: Two Frontiers

1. **Can a dual problem be more difficult to test than the original problem?**
   - Current work: Gap in upper bounds, but no separation
2. **Dual problems of graph partition problems**
   - Does testing whether a graph is far from having a large clique* reduce to tolerant testing?

* Where "large clique" means clique of density ρ|V|, for a constant predetermined ρ>0.

Thank you!

A far-from-far visual game is available at
http://sites.google.com/site/roeitell