Permanent V. Determinant: An Exponential Lower Bound Assuming Symmetry

J.M. Landsberg and Nicolas Ressayre

Texas A&M University and Univ. Lyon I

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Valiant's conjecture

Theorem (Valiant)

Let P be a homogeneous polynomial of degree m in M variables. Then there exists an n and $n \times n$ matrices $A_0, A_1, \ldots, A_M$ such that

$$P(y^1, \ldots, y^M) = \det_n(A_0 + y^1 A_1 + \cdots + y^M A_M).$$

Write $P(y) = \det_n(A(y))$.

Let $\mathrm{dc}(P)$ be the smallest n that works.

Let $Y = (y^i_j)$ be an $m \times m$ matrix and let $\mathrm{perm}_m(Y)$ denote the permanent, a homogeneous polynomial of degree m in $M = m^2$ variables.

Conjecture (Valiant, 1979)

$\mathrm{dc}(\mathrm{perm}_m)$ *grows faster than any polynomial in m.*

State of the art

$\mathrm{dc}(\mathrm{perm}_2) = 2$ (classical)

$\mathrm{dc}(\mathrm{perm}_m) \geq \frac{m^2}{2}$ (Mignon-Ressayre, 2005)

$\mathrm{dc}(\mathrm{perm}_m) \leq 2^m - 1$ (Grenet 2011, explicit expressions)

$\mathrm{dc}(\mathrm{perm}_3) = 7$ (Alper-Bogart-Velasco 2015), In particular, Grenet's representation for perm_3:

$$\mathrm{perm}_3(y) = \det{}_7 \begin{pmatrix} 0 & 0 & 0 & 0 & y_3^3 & y_2^3 & y_1^3 \\ y_1^1 & 1 & 0 & 0 & 0 & 0 & 0 \\ y_2^1 & 0 & 1 & 0 & 0 & 0 & 0 \\ y_3^1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & y_2^2 & y_1^2 & 0 & 1 & 0 & 0 \\ 0 & y_3^2 & 0 & y_1^2 & 0 & 1 & 0 \\ 0 & 0 & y_3^2 & y_2^2 & 0 & 0 & 1 \end{pmatrix},$$

is optimal.

Guiding principle: Optimal expressions should have interesting geometry

Geometric Complexity Theory principle: perm_m and $\det_n$ are special because they are determined by their *symmetry groups*:

Let $G_{\det_n}$ be the subgroup of the group of invertible linear maps $\mathbb{C}^{n^2} \to \mathbb{C}^{n^2}$ preserving the determinant, the symmetry group of $\det_n$.

For example: B, C: $n \times n$ matrices with $\det(BC) = 1$, then $\det_n(BXC) = \det_n(X)$, and $\det_n(X^T) = \det_n(X)$. These maps generate $G_{\det_n}$.

Let G_{perm_m} be the symmetry group of perm_m, a subgroup of the group of invertible linear maps $\mathbb{C}^{m^2} \to \mathbb{C}^{m^2}$.

For example, E, F: $m \times m$ permutation matrices or diagonal matrices with determinant one, then $\mathrm{perm}_m(EYF) = \mathrm{perm}_m(Y)$, and $\mathrm{perm}_m(Y^T) = \mathrm{perm}_m(Y)$. These generate G_{perm_m}.

Let $G^L_{\mathrm{perm}_m}$ be the subgroup of the group of invertible linear maps $\mathbb{C}^{m^2} \to \mathbb{C}^{m^2}$ generated by the E's.

Equivariance

Proposition (L-Ressayre)

Grenet's expressions are $G^L_{\mathrm{perm}_m}$-equivariant, namely, given $E \in G^L_{\mathrm{perm}_m}$, there exist $n \times n$ matrices B, C such that $A_{Grenet,m}(EY) = BA_{Grenet,m}(Y)C$.

For example, let

$$E(t) = \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}.$$

Then $A_{Grenet,m}(E(t)Y) = B(t)A_{Grenet,m}(Y)C(t)$, where

$$B(t) = \begin{pmatrix} t_3 & & & & & & \\ & t_1t_3 & & & & & \\ & & t_1t_3 & & & & \\ & & & t_1t_3 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix} \quad \text{and} \quad C(t) = B(t)^{-1}.$$

Main results

Theorem (L-Ressayre)

Among $G^L_{\mathrm{perm}_m}$*-equivariant determinantal expressions for* perm_m*, Grenet's size* $2^m - 1$ *expressions are optimal and unique up to trivialities.*

Theorem (L-Ressayre)

There exists a G_{perm_m}*-equivariant determinantal expression for* perm_m *of size* $\binom{2m}{m} - 1 \sim 4^m$*.*

Theorem (L-Ressayre)

Among G_{perm_m}*-equivariant determinatal expressions for* perm_m*, the size* $\binom{2m}{m} - 1$ *expressions are optimal and unique up to trivialities.*

In particular, Valiant's conjecture holds in the restricted model of equivariant expressions.

Restricted model $\rightsquigarrow$ general case?

Howe-Young duality endofunctor: The involution on the space of symmetric functions (exchanging elementary symmetric functions with complete symmetric functions) extends to modules of the general linear group.

Punch line: can exchange symmetry for skew-symmetry.

Proof came from first proving an analogous theorem for $\det_m$ (with the extra hypothesis that rank $A_0 = n - 1$) and then using the endofunctor to guide the proof.

Same idea was used in Efremeko-L-Schenck-Weyman: (i) quadratic limit of the method of shifted partial derivatives for Valiant's conjecture and (ii) linear strand of the minimal free resolution of the ideal generated by subpermanents.

More detail on the endofunctor

Idea: we know *a lot* about the determinant. Use the endofunctor to transfer information about the determinant to the permanent.
The catch: the projection operator.
Illustration:
Given a linear map $f : \mathbb{C}^n \to \mathbb{C}^n$, one obtains linear maps $f^{\wedge k} : \Lambda^k\mathbb{C}^n \to \Lambda^k\mathbb{C}^n$, whose matrix entries are the size k minors of f and whose eigenvalues are the elementary symmetric functions of the eigenvalues of f.

In particular the map $f^{\wedge n} : \Lambda^n\mathbb{C}^n = \mathbb{C} \to \Lambda^n\mathbb{C}$ is multiplication by the scalar $\det_n(f)$.

One also has linear maps $f^{\circ k} : S^k\mathbb{C}^n \to S^k\mathbb{C}^n$, whose eigenvalues are the complete symmetric functions of the eigenvalues of f.
The map $f^{\circ k}$ is Howe-Young dual to $f^{\wedge k}$.

Project $S^n\mathbb{C}^n$ to the line spanned by the square-free monomial.
The image of the map induced from $f^{\circ n}$ is the permanent.