Lower Bounds: *from* circuits *to* QBF proof systems

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Joint work with
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Leroy Chew (Leeds University)

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In this talk

a general construction for QBF proof systems

lower bounds for strong QBF proof systems

- exploit the full spectrum of circuit lower bounds via
- a new technique to transfer lower bounds

Quantified Boolean Formulas (QBF)

We consider QBFs in **prenex** form with a CNF **matrix**.

$$\text{\textit{e.g.}}\ \forall u \forall u' \exists x \exists x'\, (\neg u \vee x) \wedge (u' \vee \neg x')$$

$$\forall u \exists x\, (u \vee x) \wedge (u \vee \neg x)$$

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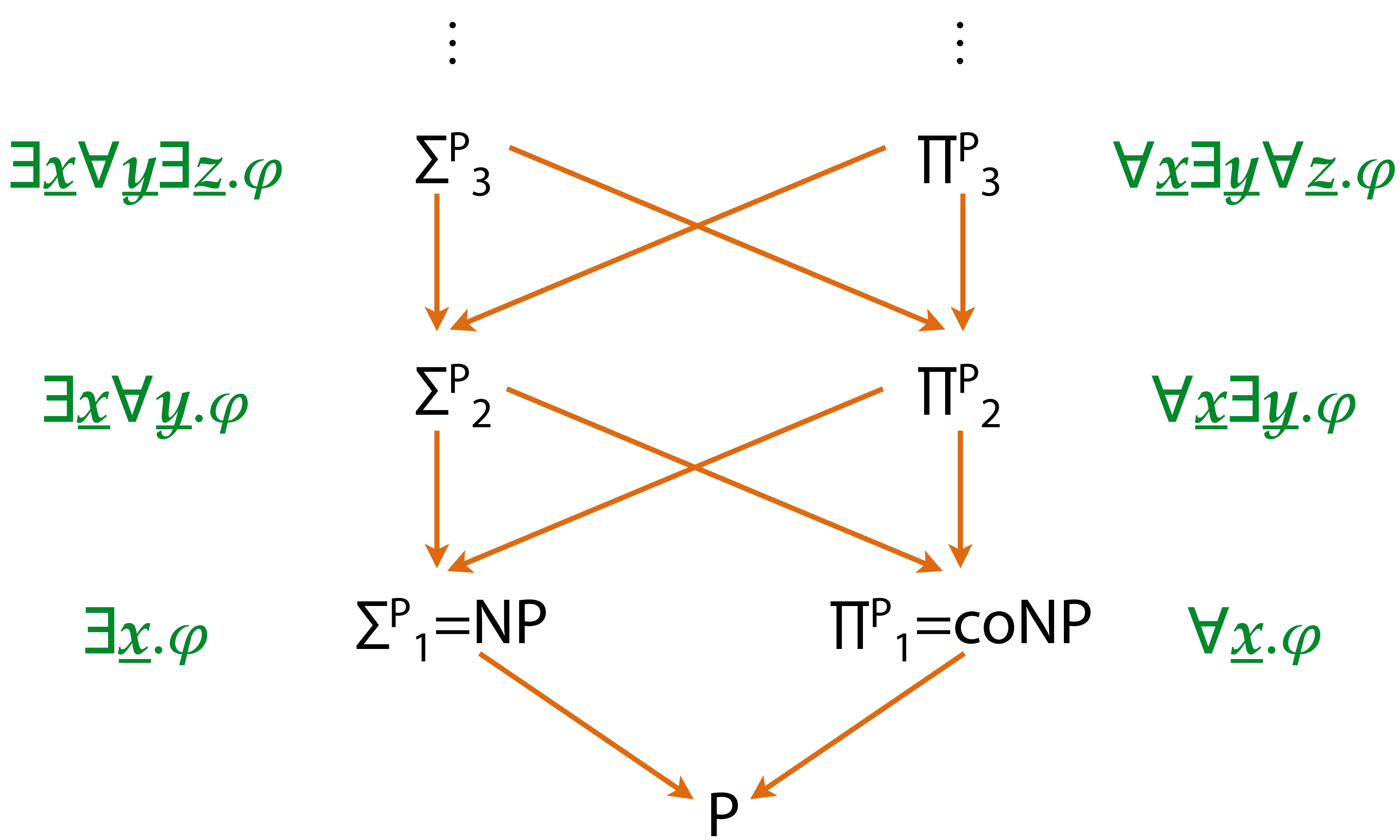

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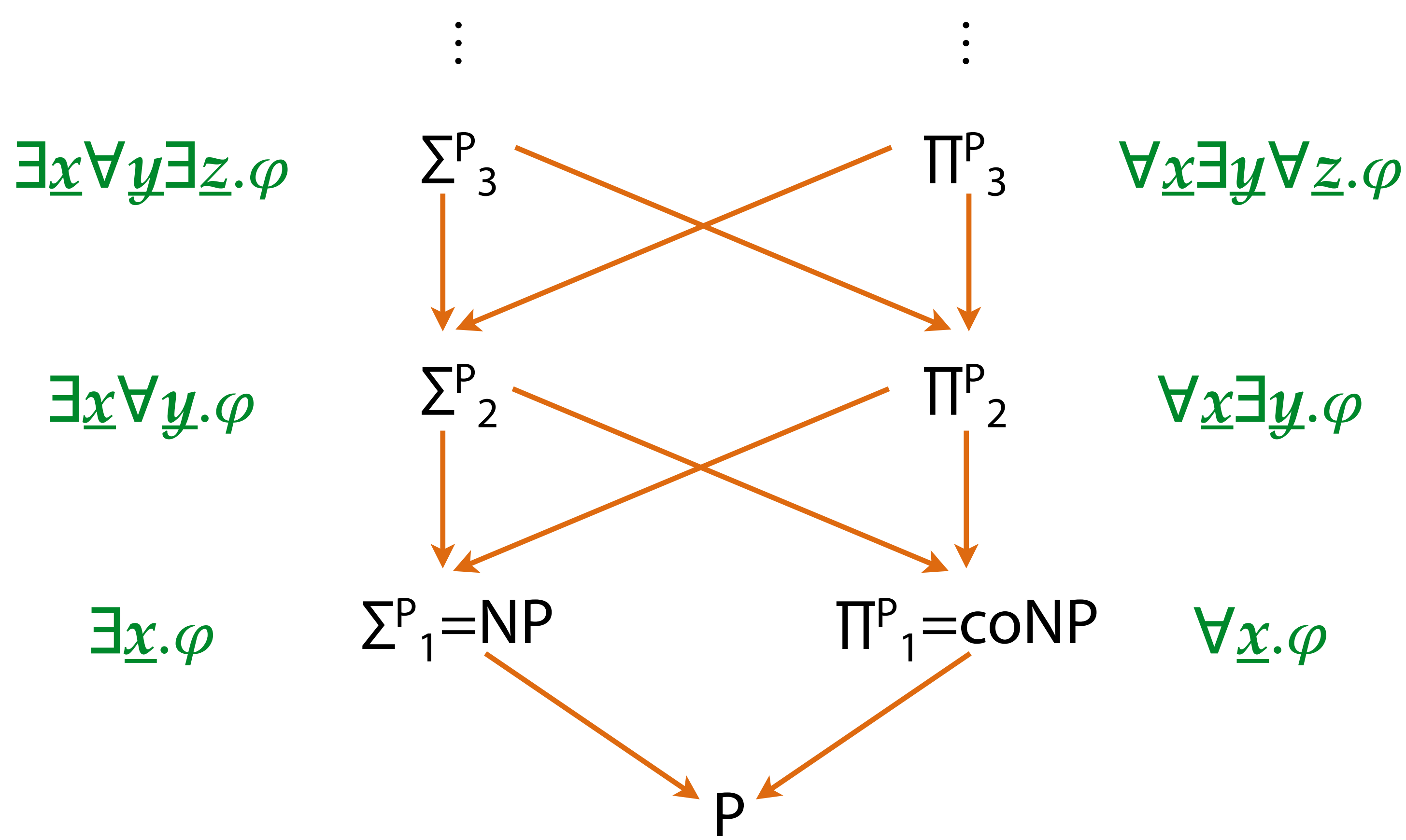

A QBF as a game between $\exists$, $\forall$

- $\exists$ and $\forall$ assign values to vars following the ordering of the prefix in the QBF
- $\exists$ / $\forall$ wins if the QBF becomes true / false
- a QBF is true / false $\iff$ exists a winning strategy for $\exists$ / $\forall$

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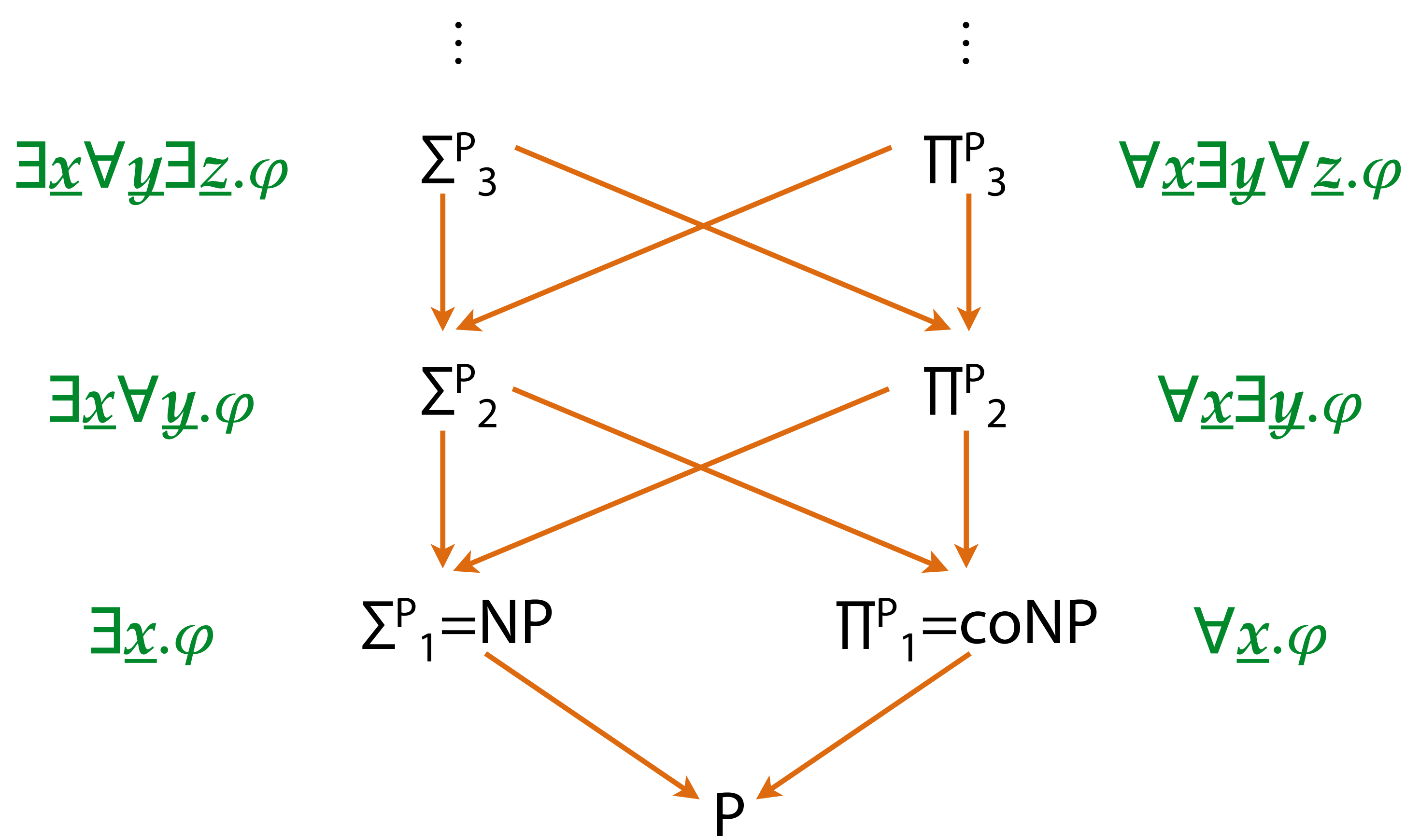
ranging over {0,1}

$\forall$ wins playing $u = 0$



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SAT / QBF solving

SAT

decide if a CNF is satisfiable

NP-complete

SAT-solvers very successful

TQBF

decide if a QBF with no free variables is true

PSPACE-complete

QBF-solvers at an early stage but they apply also to planning and verification

Theoretical tool to study performance & limitations of SAT / QBF solvers: **proof complexity!**

A **proof system** verifies if a string π is a proof of a theorem

- in poly-time wrt $|\pi|$
- it has to be sound and complete

propositional proof system = proof system for UNSAT

QBF proof system = proof system for FQBF

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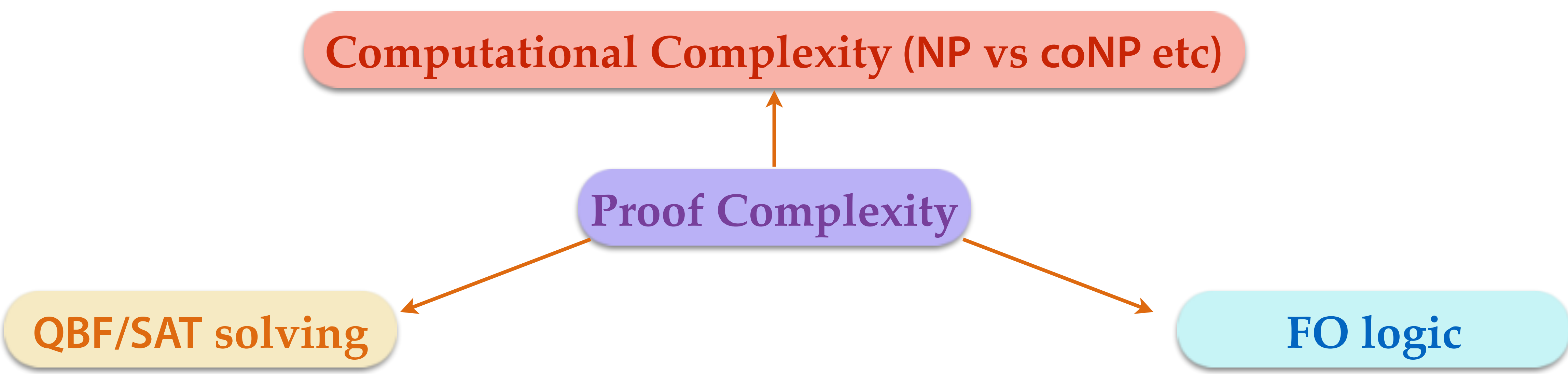

A longstanding belief

There exists a close connection between

Boolean circuits

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lower bounds for **propositional proof systems**

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not formal! (*yet*)

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this talk!

Hilbert type systems with axiom schemes (*e.g.* $A \vee \neg A$) and inference rules,

e.g. **modus ponens** $\dfrac{A \quad A \longrightarrow B}{B}$

$\mathcal{C}$-FREGE systems

The circuit class $\mathcal{C}$ restricts the formulas allowed in the system

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$AC^0[p]$-FREGE = bounded depth FREGE with MOD_p gates

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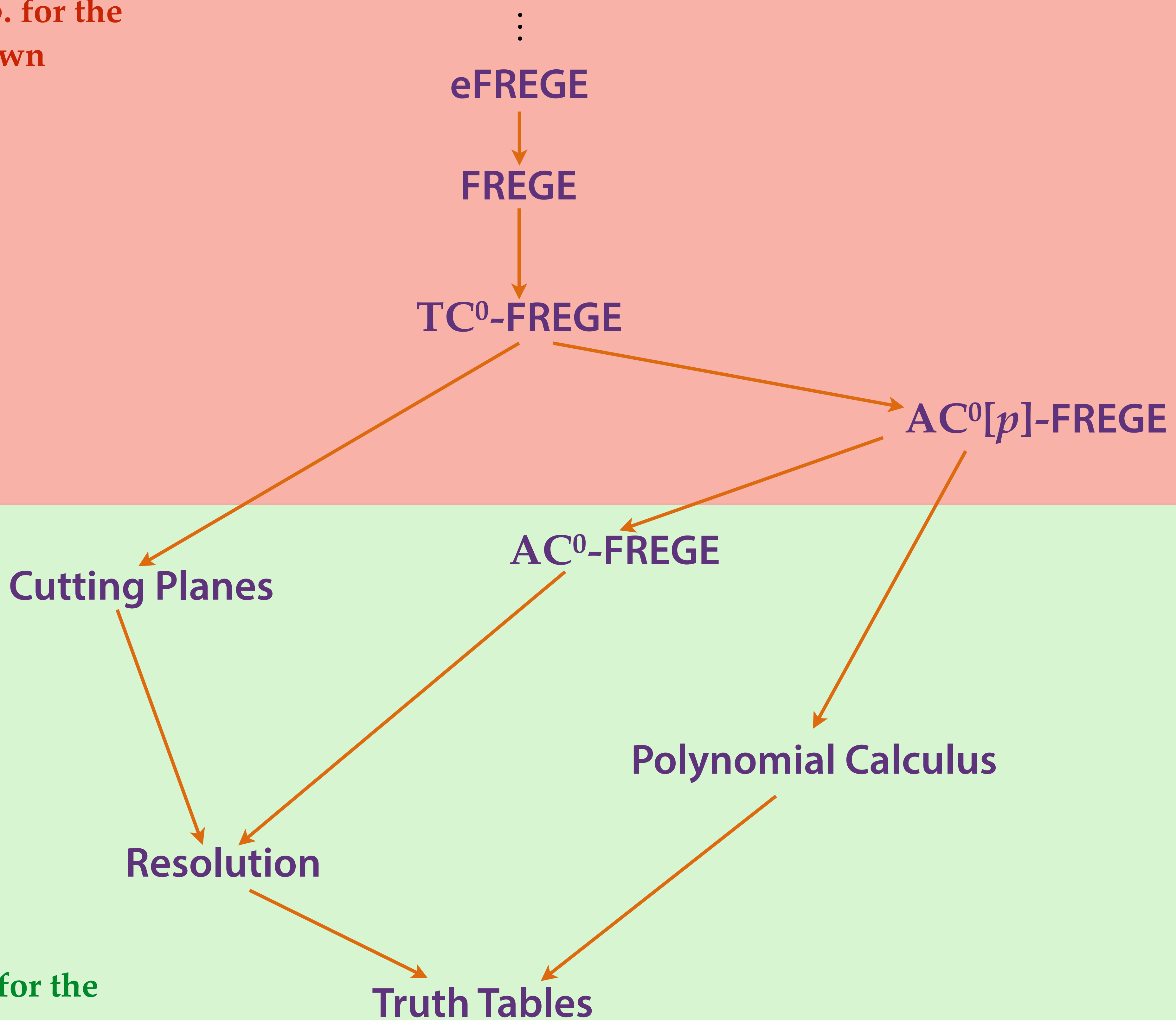
TC^0-FREGE = bounded depth FREGE with threshold gates

A lattice of proof systems

<u>no</u> superpolynomial l.b. for the size of proofs known



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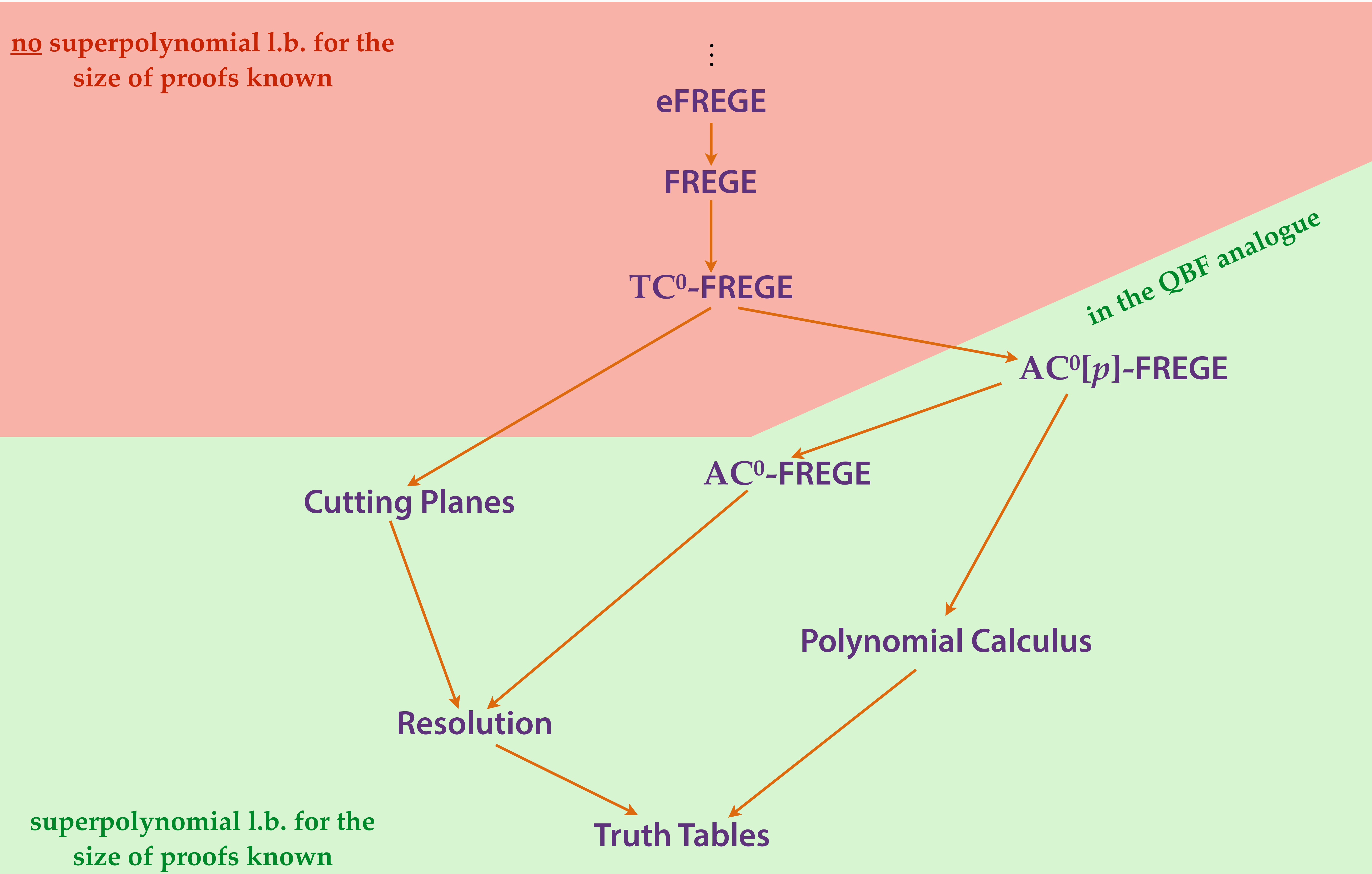
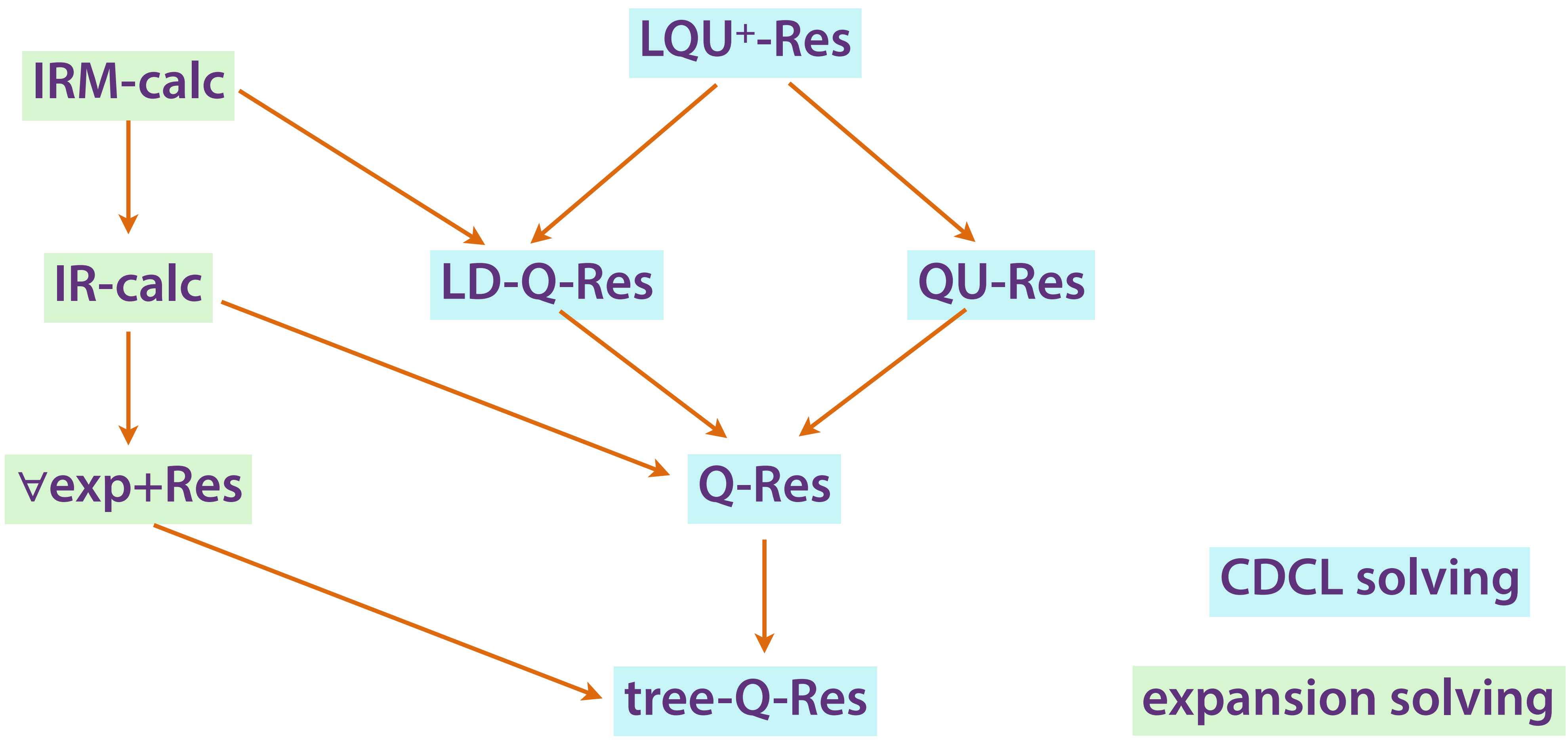


QBF proof systems



- no unique analogue of Resolution
- various sequent calculi exists as well

[Krajicek,Pudlak '00; Cook,Morioka '05; Egli '12]

- some of the techniques used in Resolution transfer to "QBF Resolution" (*e.g.* interpolation) some don't (*e.g.* size-width relationship)

[Beyersdorff, Chew, Mahajan, Shukla ICALP'15 & STACS'16]

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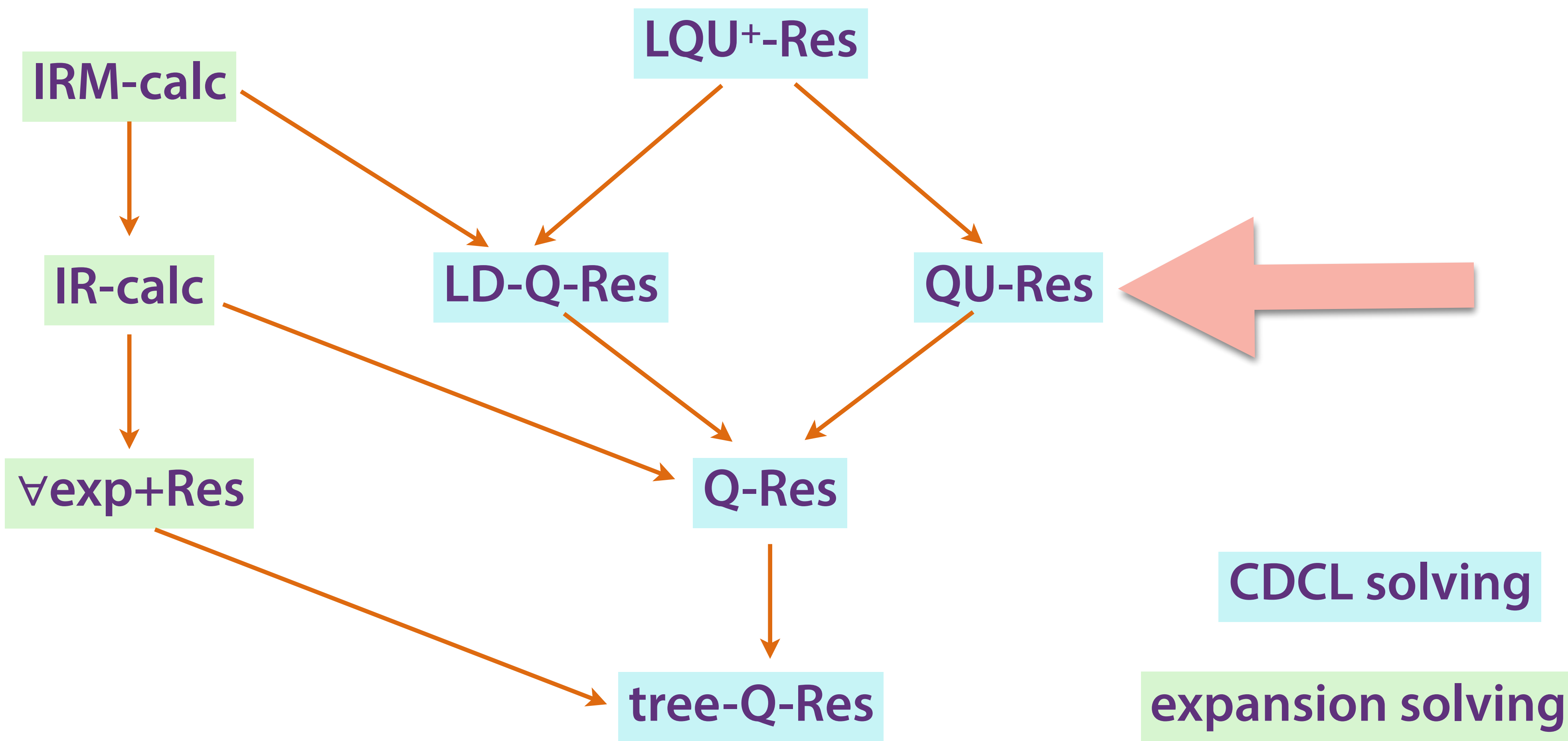

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RES+∀red (= QU-Res)

the usual inference rule of Resolution

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$$\frac{C}{C\restriction_{u=0}} \qquad \frac{C}{C\restriction_{u=1}}$$ where u is **universal** & innermost among the vars of C

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e.g. $\forall u \exists x\ (u \vee x) \wedge (u \vee \neg x)$ $(u \vee x)$ $(u \vee \neg x)$

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$$\begin{tikzcd}
(u \lor x) \arrow[dr] & & (u \lor \neg x) \arrow[dl] \\
& u \arrow[d] & \\
& \bot &
\end{tikzcd}$$

$\mathcal{C}$-FREGE+∀red has

- the inference rules of $\mathcal{C}$-FREGE &
- a ∀red rule:

$$\frac{L}{L[u/B]}$$

where (1) u is **universal** & innermost among the vars of L

(2) $L[u/B]$ belongs to $\mathcal{C}$ & B contains only vars on the left of u in the prefix Q of the false QBF Q.φ to be refuted

$\mathcal{C}$-FREGE+∀red is sound and complete for QBF

How to prove lower bounds?

- every false QBF has a winning strategy for ∀
- *(hope)* hard strategies require large proofs

  ≡ *short proofs lead to easy strategies*
- find false QBFs such that every strategy for ∀ is hard to compute

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Strategy Extraction Theorem

Given a false QBF $Q.\varphi$ and a refutation π of it in $\mathcal{C}$-**FREGE+∀red**
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this generalize an analogous result for Q-RES by [Balabanov,Jiang '12]

From functions to QBFs

Let $f(\underline{\boldsymbol{x}})$ be a Boolean function, **Q-f** is the following QBF

$$\textbf{Q-}f \equiv \exists \underline{\boldsymbol{x}} \forall u \exists \underline{\boldsymbol{t}}.\ u \nleftrightarrow f(\underline{\boldsymbol{x}})$$

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The only winning strategy for $\forall$ to win **Q-f** is to play $u \leftarrow f(\underline{x})$

e.g. **Q-parity** $= \exists\, x_1,\ldots,x_n \forall u \exists \underline{t}.\ u \not\leftrightarrow x_1 \oplus \cdots \oplus x_n$

$$= \exists\, x_1,\ldots,x_n \forall u \exists\, t_2,\ldots,t_n.\ (u \not\leftrightarrow t_n) \wedge (t_2 \leftrightarrow x_1 \oplus x_2)$$

$$\wedge \ldots$$

$$\wedge (t_i \leftrightarrow t_{i-1} \oplus x_i)$$

$$\wedge \ldots$$

$$\wedge (t_n \leftrightarrow t_{n-1} \oplus x_n)$$

A lower bound for $AC^0[p]$-FREGE+∀red

For each prime $p \neq 2$, **Q-parity** require exponential size **$AC^0[p]$-FREGE+∀red** proofs

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Proof (sketch).

- by contradiction, let π be a poly-size refutation of **Q-parity** in **$AC^0[p]$-FREGE+∀red**
- By the Strategy Extraction Theorem we obtain from π a poly-size **$AC^0[p]$**-circuit computing **parity**
- By [Razborov,Smolensky '87] **parity** needs exponential size **$AC^0[p]$**-circuits ☠️

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this approach was used for **Q-Res** *by* [Balabanov,Jiang '12; Beyersdorff, Chew,Janota'15]

Separations

There exists a QBF that has poly-size proofs in **depth *d*-Frege+∀red** &
requires proofs of exponential size in **depth (*d*-3)-Frege+∀red**

p,q distinct primes, there exists a QBF that

- require exponential size proofs in **$AC^0[p]$-Frege+∀red**
- have poly-size proofs in **$AC^0[q]$-Frege+∀red**

TC^0-Frege+∀red is exponentially stronger than **$AC^0[p]$-Frege+∀red**

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no separation known with formulas
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we use Q-Sipser$_d$ where Sipser$_d$ esponentially separates *depth d* from *depth* (*d*-1) circuits [Hastad '86]

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carefully encoding Q-MOD$_q$ & [Smolensky '87] lower bound

propositional case:
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TC^0-Frege+∀red is exponentially stronger than **AC^0[p]-Frege+∀red**

carefully encoding Q-majority & [Razborov-Smolensky '87] lower bound

propositional case:
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Conditional lower bounds

If PSPACE $\not\subseteq$ NC^1 then there exists a false QBF requiring super-polynomial size refutations in **Frege+∀red**

If PSPACE $\not\subseteq$ P/$_{\text{poly}}$ then there exists a false QBF requiring super-polynomial size refutations in **eFrege+∀red**

¿(*Unconditional*) Size lower bounds for **Frege+∀red**?

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Thanks!

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