Weighted Gate Elimination

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Gate Elimination

Lower Bounds for Affine Dispersers

Lower Bound for Quadratic Dispersers

Open Problems

Outline

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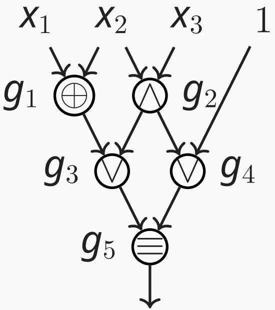
Lower Bound for Quadratic Dispersers

Open Problems

Boolean Circuits

Inputs:
$x_1, \ldots, x_n, 0, 1$
Gates:
binary functions
Fan-out:
unbounded
Depth:
unbounded

$$
\begin{aligned}
g_1 &= x_1 \oplus x_2 \\
g_2 &= x_2 \wedge x_3 \\
g_3 &= g_1 \vee g_2 \\
g_4 &= g_2 \vee 1 \\
g_5 &= g_3 \equiv g4
\end{aligned}
$$



Exponential Bounds

Lower Bound

Counting shows that almost all functions of n variables have circuit size $\Omega(2^n/n)$ [Shannon 1949].

Upper Bound

Any function can be computed by circuits of size $(1+o(1))2^n/n$ [Lupanov 1958].

Explicit Lower Bounds

Previous

$2n$	$f(x) = \bigoplus_{i<j} x_i x_j$	[Kloss, Malyshev 1965]
$2n$	$f(x) = [\sum x_i \equiv_3 0]$	[Schnorr 1974]
$2.5n$	$f(x, a, b) = x_a \oplus x_b$	[Paul 1977]
$2.5n$	symmetric	[Stockmeyer 1977]
$3n$	$f(x, a, b, c) = x_a x_b \oplus x_c$	[Blum 1984]
$3n$	affine dispersers	[Demenkov, Kulikov 2011]
$3.011n$	affine dispersers	[FGHK 2015]

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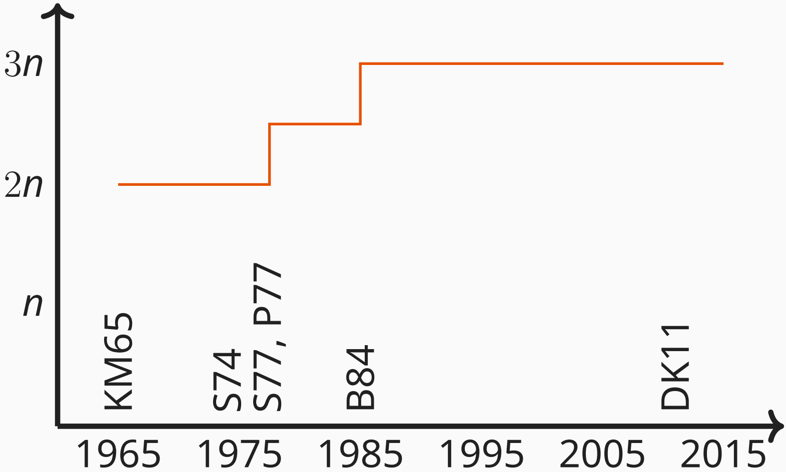
$3n$	affine dispersers	[Demenkov, Kulikov 2011]
$3.011n$	affine dispersers	[FGHK 2015]

New

$3.11n$	quadratic dispersers (non-explicit)	[G, Kulikov 2015]

Explicit Lower Bounds: Pictorially



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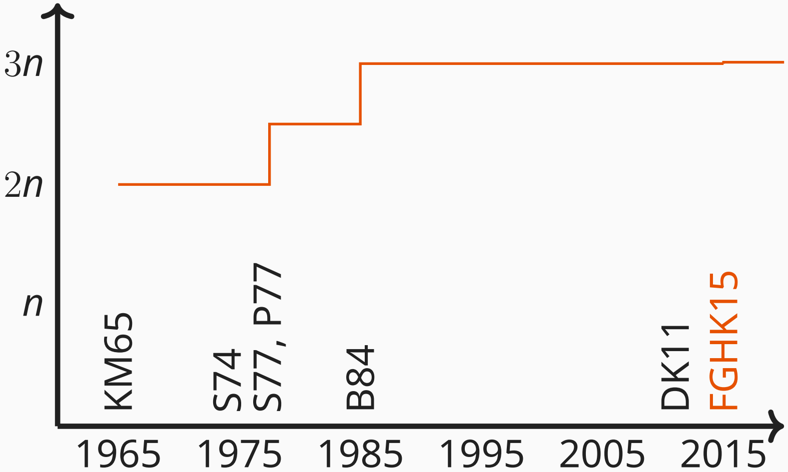

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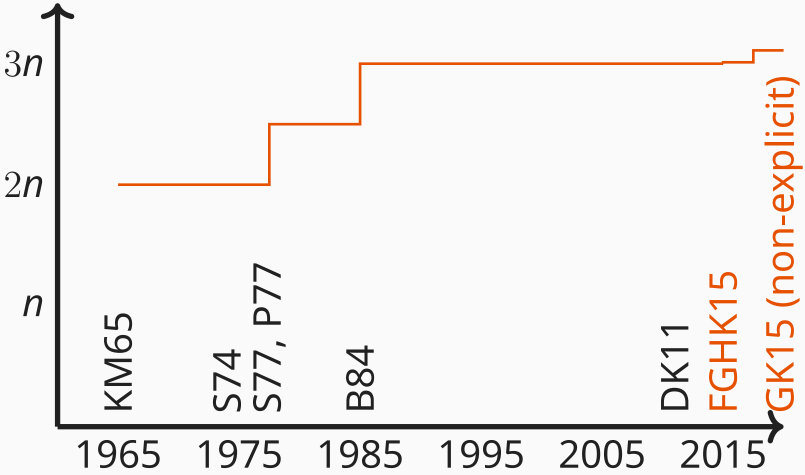
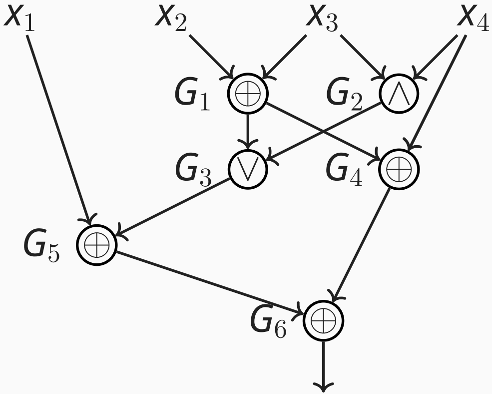


Gate Elimination Method

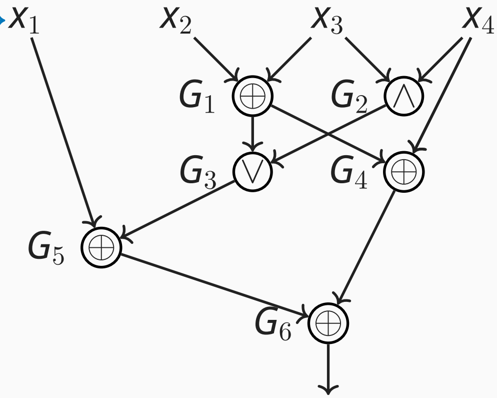
To prove, say, a $3n$ lower bound for all functions f from a certain class $\mathcal{C}$:

- show that for any circuit computing f, one can find a substitution eliminating at least 3 gates;
- show that the resulting subfunction still belongs to $\mathcal{C}$;
- proceed by induction.

Gate Elimination: Example

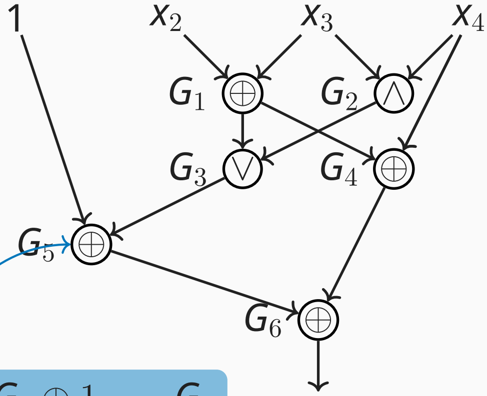


Gate Elimination: Example



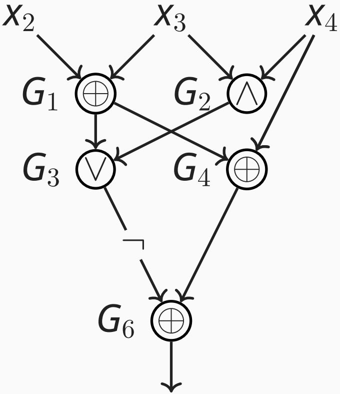
assign $x_1 = 1$

Gate Elimination: Example

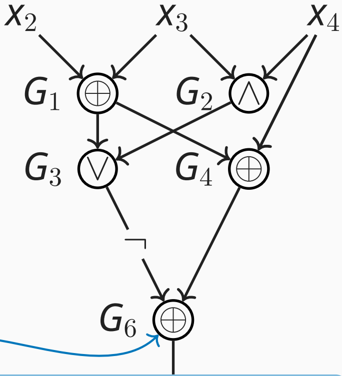


G_5 now computes $G_3 \oplus 1 = \neg G_3$

Gate Elimination: Example

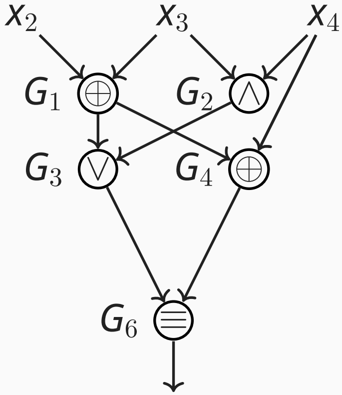


Gate Elimination: Example



now we can change the binary function assigned to G_6

Gate Elimination: Example



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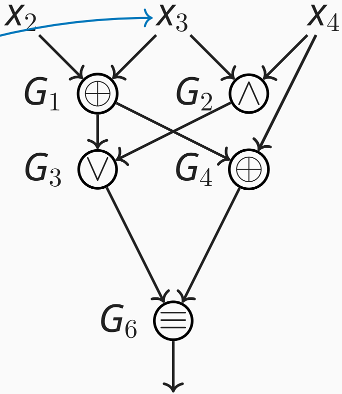
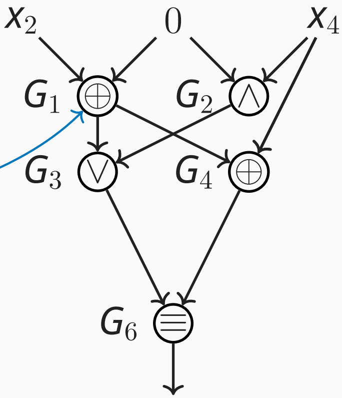


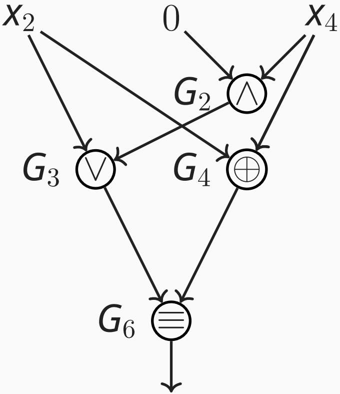
now assign $x_3 = 0$

Gate Elimination: Example

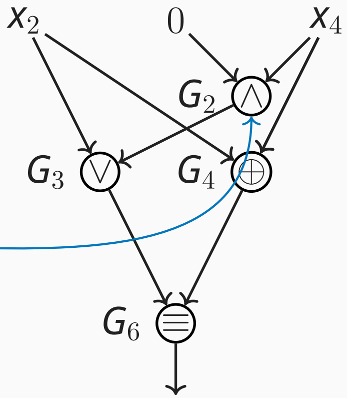


G_1 then is equal to x_2

Gate Elimination: Example

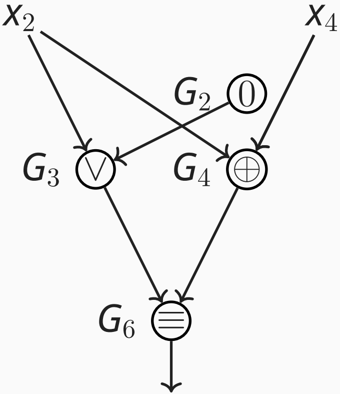


Gate Elimination: Example

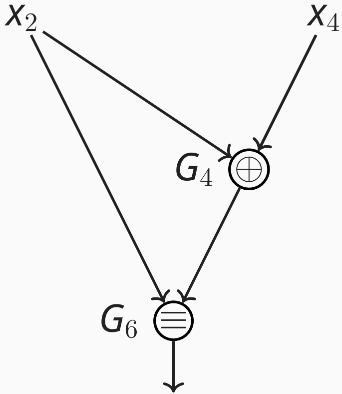


$G_2 = 0$

Gate Elimination: Example



Gate Elimination: Example



Binary Functions

There are 16 Boolean functions.

- 2 constant functions: 0, 1;
- 4 degenerate functions: x, $x \oplus 1$, y, $y \oplus 1$;
- 2 xor-type functions: $x \oplus y$, $x \oplus y \oplus 1$;
- 8 and-type functions: $(x \oplus a)(y \oplus b) \oplus c$ where $a, b, c \in 0, 1$.

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Lower Bound for Quadratic Dispersers

Open Problems

$3n$ Lower Bound

Theorem

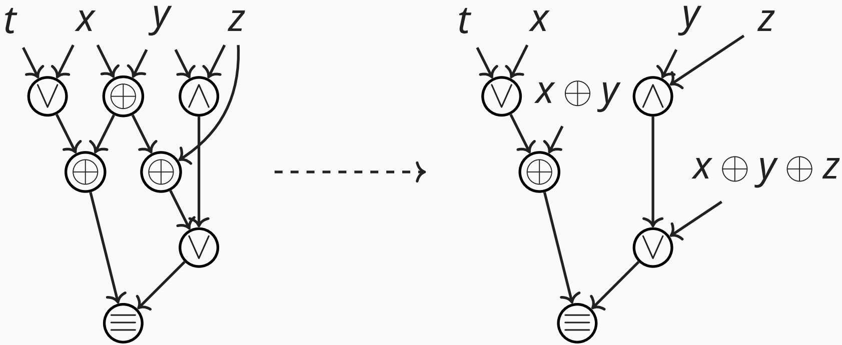
If $f\colon \{0,1\}^n \to \{0,1\}$ is an affine disperser for dimension $d = o(n)$, then

$$\mathrm{size}(f) \geq 3n - o(n).$$

Affine Dispersers

- A function $f\colon \{0,1\}^n \to \{0,1\}$ is called an affine disperser for dimension d if it is non-constant on any affine subspace of dimension at least d.
- An affine dispereser for dimension d cannot become constant after any $n - d$ linear substitutions (i.e., substitutions of the form $x_2 \oplus x_3 \oplus x_9 = 0$).
- There exist explicit constructions of affine dispersers for subliner dimension $d = o(n)$ (e.g., [Ben-Sasson, Kopparty, 2012]).

XOR-layered Circuits



$\text{inputs}(\mathcal{C}) = 4$
$\text{size}(\mathcal{C}) = 7$

$\text{inputs}(\mathcal{C}') = 6$
$\text{size}(\mathcal{C}') = 5$

$$\text{inputs}(\mathcal{C}') + \text{size}(\mathcal{C}') \leq \text{inputs}(\mathcal{C}) + \text{size}(\mathcal{C}).$$

$3n - o(n)$ Lower Bound

Theorem [Demenkov, Kulikov 2011]

For a circuit C computing an affine disperser for dimension d:

$$\mathrm{inputs}(\mathcal{C}) + \mathrm{size}(\mathcal{C}) \geq 4(n - d)\,.$$

Corollary

$\mathrm{size}(f) \geq 3n - o(n)$ for an affine disperser for $d = o(n)$.

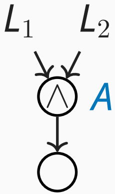
Proposition

The bound is tight: $\mathrm{size}(IP) = n - 1$ and IP is an affine disperser for dimension $d = n/2 + 1$.

Proof

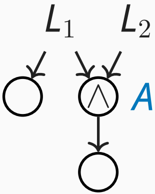
- Want to show:
$$\text{inputs}(\mathcal{C}) + \text{size}(\mathcal{C}) \geq 4(n-d).$$
- Make $n-d$ affine restrictions each time reducing $(\text{inputs} + \text{size})$ by at least 4.
- Convert $\mathcal{C}$ to XOR-layered and take a top-gate A:

Case 1



$L_1 \leftarrow 0$:
$\Delta\,\text{size} = 2$
$\Delta\,\text{inp} = 2$

Case 2



$L_1 \leftarrow 0$:
$\Delta\,\text{size} = 3$
$\Delta\,\text{inp} = 1$

Outline

Main Result

Theorem

Let $f\colon \{0,1\}^n \to \{0,1\}$ *be a function that is not constant on any set* $S \subseteq \{0,1\}^n$ *of size at least* $2^{n/100}$ *that can be defined as*

$$S = \{x\colon p_1(x) = \cdots = p_{2n}(x) = 0\},\ \deg(p_i) \leq 2.$$

Then

$$\mathrm{size}(f) \geq 3.11n\,.$$

Quadratic Dispersers

- A random function is not constant on any set S of size s that can be defined as

$$S = \{x \in \{0,1\}^n : p_1(x) = \cdots = p_{s/n^3}(x) = 0\}.$$

- We need much weaker dispersers: $(n, 2n, 2^{n/100})$-dispersers. Even in NP. Even with multiple outputs.

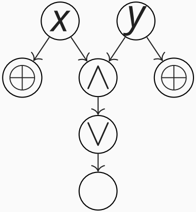
Regular Gate Elimination

- make a substitution;
- decrease S by a factor of 2;
- eliminate at least 3 gates;
- S belongs to the same class;
- repeat $n - o(n)$ times.

Weighted Gate Elimination

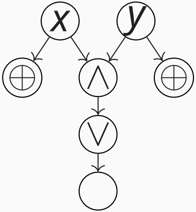
- make a restriction;
- decrease *S* by a factor of α;
- make sure to eliminate at least $3\log\alpha$ gates;
- *S* belongs to the same class;
- repeat until *S* becomes small (e.g., $2^{n/100}$).

Toy Example

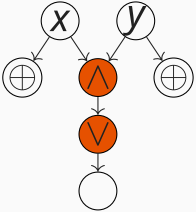


$$S \subseteq \{0, 1\}^n$$

Toy Example



$$xy = 0$$



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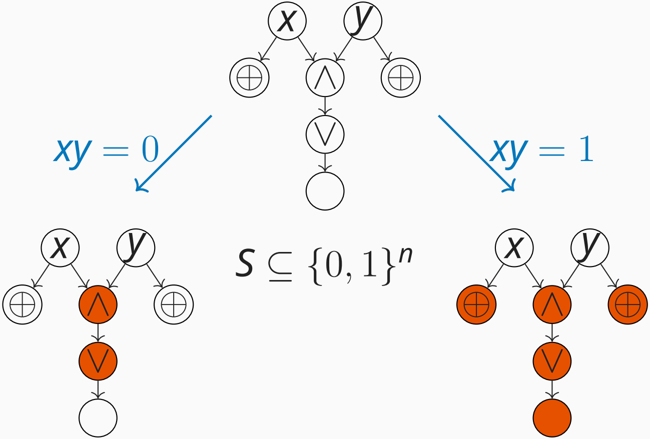

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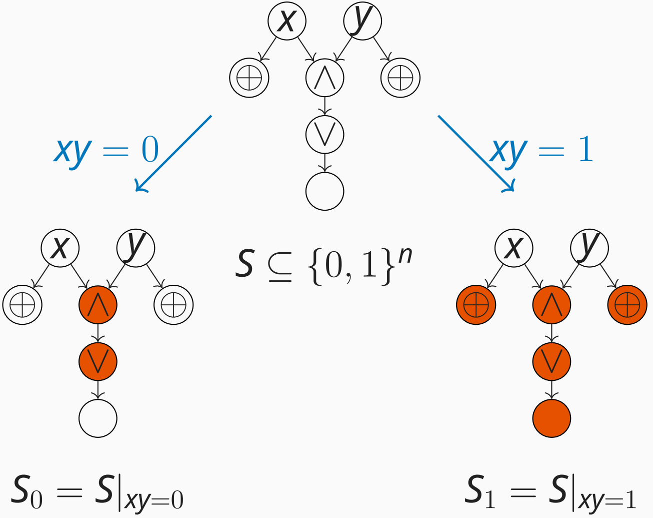

x y
$\oplus$ $\wedge$ $\oplus$
$\vee$
$xy = 0$
$xy = 1$
$S \subseteq \{0,1\}^n$
$S_0 = S|_{xy=0}$
$S_1 = S|_{xy=1}$

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- Quadratic dispersers in NP?

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- Quadratic dispersers in NP?
- Lower bounds in other models?

Open Problems

- Quadratic dispersers in NP?
- Lower bounds in other models?
- Connections to algorithms for Circuit-SAT?

Thank you for your attention!