On Hardness of Approximating the Parameterized Clique Problem

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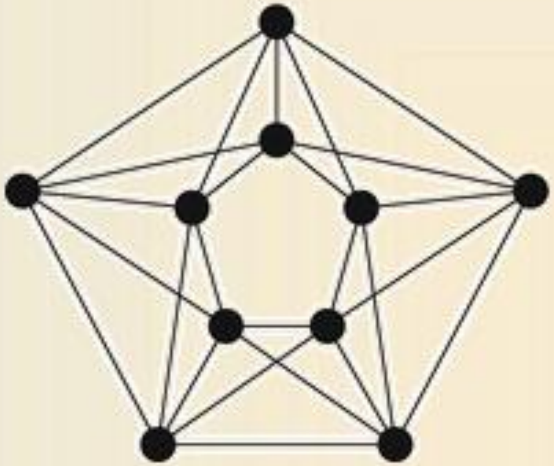
The clique problem

The Clique problem:

Input: A graph $G=(V,E)$ on n vertices, and a parameter k.

Goal: Find a k-clique in G (or declare ”there is no k-clique”).



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[FGLSS ‘96]: It is NP-hard to find a clique of size k/2.

[Håstad ‘99]: For $k=n^{0.99}$ it is NP-hard to find a clique of size $n^{0.01}$.

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Well, what can I say?
Looks like a very hard problem…

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The parameterized k-Clique problem:

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Is the k-Clique problem *fixed-parameter tractable*?

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Can we hope for something similar for the **k-*Clique*** problem?

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Can we hope for something similar for the ***k-Clique*** problem?

Assuming ETH, **k-Clique** cannot be solved in time $f(k) \cdot poly(n)$.

Approximating the Clique problem

Gap-Clique(k, k/2) problem:

Input: A graph $G=(V,E)$ on n vertices.
Goal: Decide between:

- YES case: G contains a k-clique.
- NO case: G contains no clique of size $k/2$-clique.

Question: Can we solve **Gap-Clique** in time $f(k) \cdot \mathrm{poly}(n)$?

Is the Gap-Clique problem *fixed-parameter tractable*?

Main Result

In the paper we give *evidence* that
Gap-Clique(k, k/2) is ***not*** *fixed-parameter tractable*.

We define a constraint satisfaction problem called k-DEG-2-SAT,
and show an FPT-reduction

$$\text{k-DEG-2-SAT} \leq_{FPT} \textit{Gap-Clique}(k, k/2)$$

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Definition: [$A \leq_{FPT} B$]

An FPT-reduction from **A** to **B**

gets an instance **(x,k)** of **A** and outputs an instance **(x',k')** of **B**

such that

1. **(x,k) ∈ A** if and only if **(x' ,k') ∈ B**

2. **k'** depends only on **k.**

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If $A \leq_{FPT} B$ and B has a FPT-algorithm,

then A also has an FPT-algorithm .

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Caveat: We do not know the status
of the k-DEG-2-SAT problem.

Could be fixed-parameter tractable …

The k-DEG-2-SAT problem

The k-DEG-2-SAT problem:

Input: A finite field **F** of size n, and a system of k quadratic equations over **F** in k variables $x_1,\ldots x_k$.

$$p_1(x_1,\ldots x_k)=0, \quad \ldots \quad p_k(x_1,\ldots x_k)=0.$$

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Fact: ***k-DEG-2-SAT*** is NP-complete.

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Some observations:

1. There is a trivial algorithm with running time $O(n^k)$.
2. Using Gröbner bases it is possible to find a solution in the *extension field* of **F** in FPT-time.

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Note: For each n there are $n^{poly(k)}$ instances of size n.

Doesn't seem to rule out hardness for FPT-algorithms.

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Use algebraic techniques from the proof of the PCP theorem [AS, ALMSS, FGLSS, LFKN, BLR]

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- BLR linearity testing/self correcting
- FGLSS reduction

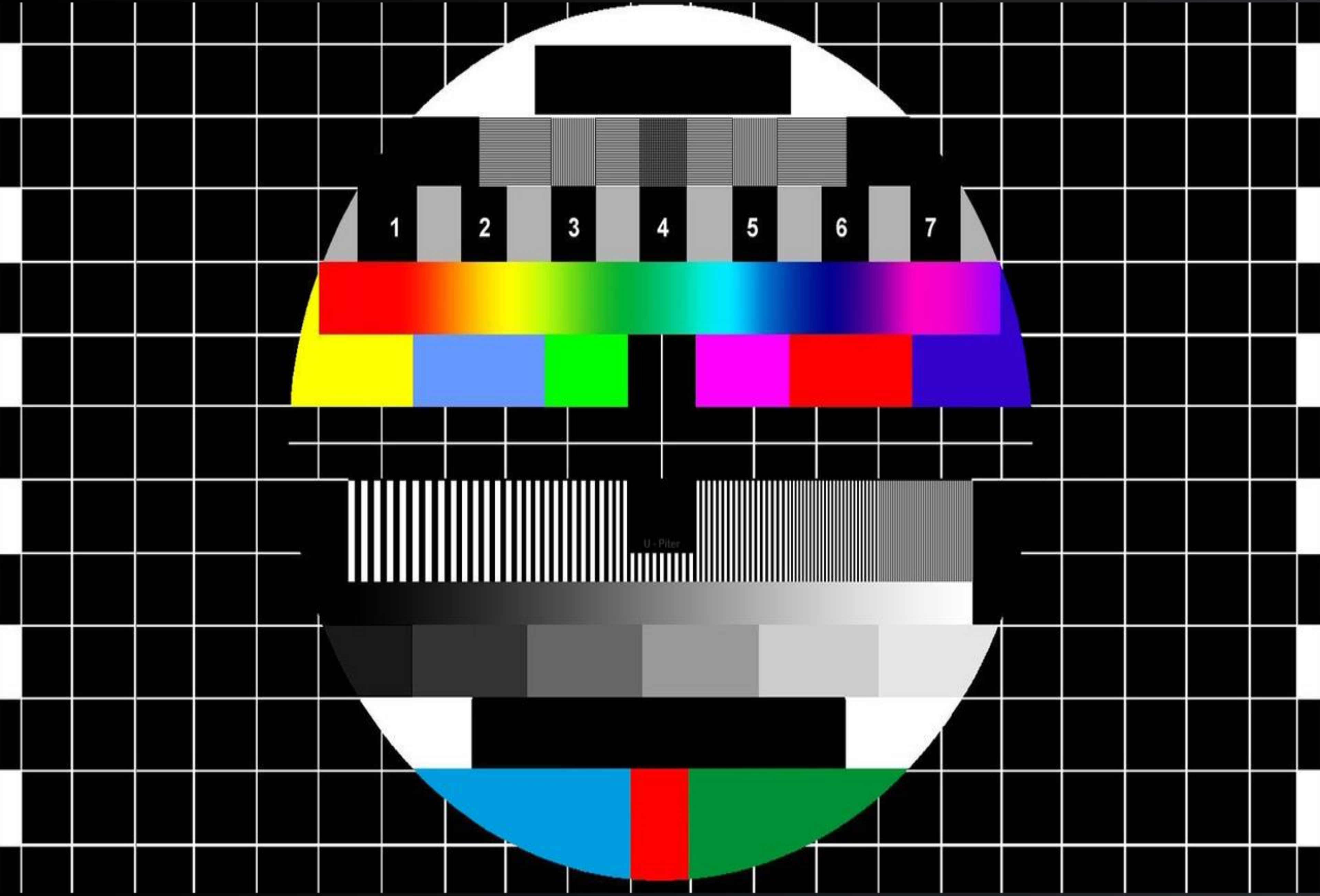
Open problems

1. Give more evidence that *Gap-Clique(k, k/2)* is not fixed-parameter tractable.
   (*Ideally*: show $k\text{-Clique} \leq_{FPT} \text{Gap-Clique}(k, k/2)$)

2. Show $\text{Gap-Clique}(k, k/2) \leq_{FPT} \text{Gap-Clique}(k, k^{0.9})$.

3. Is *Gap-Clique(k, loglog(k))* fixed-parameter tractable?

Thank You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