Computational Complexity of Piano-Hinged Dissections

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Abstract

We prove NP-completeness of deciding whether a given loop of colored right isosceles triangles, hinged together at edges, can be folded into a specified rectangular three-color pattern. By contrast, the same problem becomes polynomially solvable with one color or when the target shape is a tree-shaped polyomino.

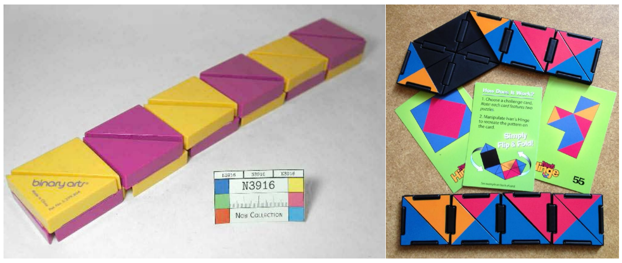

Figure 1: GeoLoop (left) and Ivan's Hinge (right)

1 Introduction

One of the simplest and most practical physical folding structures is that of a *hinge*, as in most doors or attaching the lid to a grand piano. Frederickson [4] introduced a way to make folding structures out of such hinges that can change their shape between "nearly 2D" shapes. The basic idea is to thicken a (doubly covered) 2D polygon by extruding it orthogonally into a height-2ϵ 3D prism, divide that prism into two height-ϵ layers, further divide those layers into ϵ-thickened polygonal pieces, and hinge the pieces together with hinges along shared edges. The goal in a *piano-hinged dissection* is to find a connected hinging of ϵ-thickened polygonal pieces that can fold into two (or more) different 2ϵ-thickened polygons.

Piano-hinged dissections are meant to be a more practical form of *hinged dissections*, which typically use point hinges and thus are more difficult to build [4]. Although hinged dissections have recently been shown to exist for any finite set of polygons of equal area [1], no such result is known for piano-hinged dissections.

Here we study a family of simple piano-hinged dissections, which we call a *piano-hinged loop*: $4n$ identical ϵ-thickened right isosceles triangles, alternating in orientation, and connected into a loop by hinges on the bottoms of their isosceles sides; see Figure 2. Frederickson [4, chapter 11] mentions without proof that this piano-hinged dissection can fold into any (2ϵ-thickened) n-omino, that is, any connected edge-to-edge joining of n unit squares.

Two commercial puzzles, shown in Figure 1, consist of piano-hinged loops [4]. *GeoLoop* is a piano-hinged loop with $n = 6$ that was patented by Kenneth Stevens in 1993/1994 [6] and sold by Binary Arts[1] in 1996. The pieces alternate between two colors, and by a checkerboard property of the piano-hinged loop, the resulting squares of any polyomino will alternate in color (on either side), so this puzzle is effectively *uncolored*. *Ivan's Hinge* is a piano-hinged loop with $n = 4$ that was patented by Jan Essebaggers and Ivan Moscovich in 1993/1994 [3] and sold by Paradigm Games in the mid-to-late 1990s [4] and recently by Fat Brain Toys[2]. Each piece is colored irregularly with one of two colors, and the goal in this puzzle is to make not only the specified tetromino shape but also the specified two-color pattern.

Our results. In this paper, we investigate the computational complexity of folding colored and uncolored piano-hinged loop puzzles into n-ominoes.

First we consider the uncolored piano-hinged loop, as in GeoLoop. For completeness, we prove Frederickson's claim that this loop can realize any 2ϵ-thickened n-omino, by mimicking a simple inductive argument for hinged dissections of polyominoes from [2]. For the special case of *tree-shaped* polyominoes, where the dual graph of edge-to-edge adjacencies among unit squares forms a tree, we prove further that the folding of the piano-hinged loop is unique up to cyclic shifts of the pieces in the loop.

Next we consider colored piano-hinged loops, as in Ivan's Hinge. For tree-shaped polyominoes, the previous uniqueness result implies that the problem can be solved in $O(n^2)$ time by trying all cyclic shifts. (In

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[1]Binary Arts changed its name to ThinkFun (http://www.thinkfun.com) in 2003.

[2]http://www.fatbraintoys.com

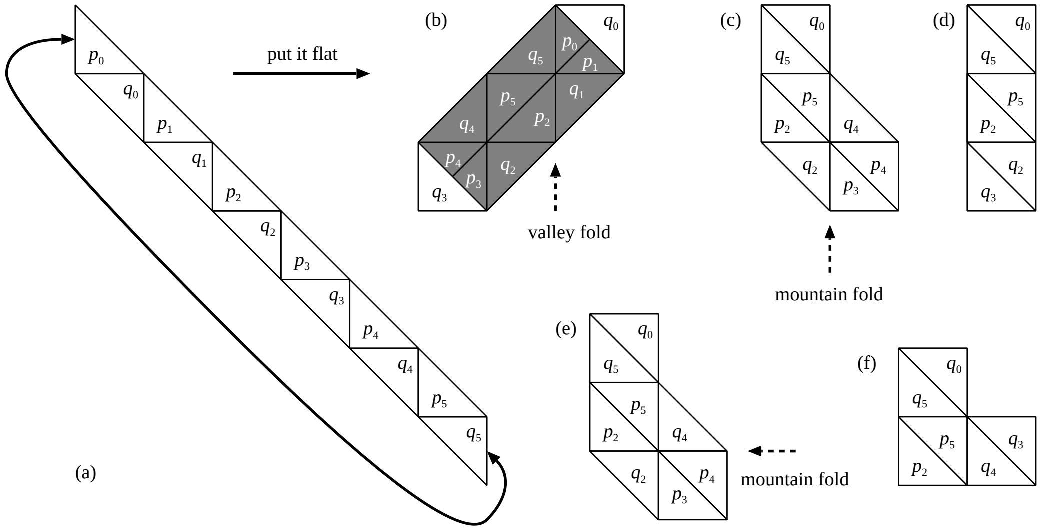

Figure 2: Piano-hinged loop with $n = 3$ (a), and two sequences of folding operations that result in a doubly covered straight tromino (b, c, d) or L tromino (b, e, f). Gray denotes the back side of the pieces.

particular, this observation makes the $n = 4$ case of Ivan's Hinge easy to solve in practice, as each tetromino has either 1 or 4 spanning trees to try.) For general polyominoes, we prove that the problem is NP-complete even if the number of colors is 3, each piece is colored uniformly one color, and the target shape is a rectangle.

Finally, we consider a related paper-folding problem: given a (monochromatic) sheet of paper of arbitrary shape, can it fold into a doubly covered polyomino whose silhouette is a specified rectangle? This seemingly simple problem is also NP-complete.

2 Preliminaries

A *piano-hinged loop* consists of a loop of $4n$ consecutive isosceles right triangles $p_0, q_0, p_1, q_2, \ldots, p_{2n-1}, q_{2n-1}$, as shown in Figure 2. Every two consecutive triangular pieces share one of two isosceles edges. The p_i's have a common orientation (collinear hypotenuses when unfolded), as do the q_i's, and these two orientations differ from each other. Each shared edge is a *piano hinge* on the back side that permits bending inward (bringing the two back sides together).

In a *folded state* of the piano-hinged loop into a doubly covered polyomino, (1) each piano hinge is flat (180°) or folded inward (360°); and (2) each unit square of the polyomino consumes exactly four triangles, with two triangles on the front and two on the back side. Thus, in any folded state, the exposed surface consists of all front sides of the pieces, while the back sides of all pieces remain hidden on the inside. Therefore, we can ignore the color of the back side of each piece, so for simplicity we can assume that each piece has a uniform color (instead of a different color on each side). Let $c(p_i)$ and $c(q_i)$ denote the color of piece p_i and q_i.

For the resultant polyomino P of n unit squares, we define the *connection graph* $G(P) = (V, E)$ as follows: V consists of n unit squares, and E contains an edge $\{u, v\}$ if and only if squares u and v are adjacent (share an edge) in P. Having $\{u, v\} \in E$ is a necessary but not sufficient condition for there to be a hinge connecting the four pieces representing square u to the four pieces representing square v; if there is such a hinge, we call u and v *joined*.

The *uncolored piano-hinged loop problem* asks whether a given polyomino can be constructed as (the silhouette of) a folded state of a given piano-hinged loop. The "silhouette" phrasing allows the folding to have unjoined squares, which are adjacent in the polyomino but not attached by a hinge in the folded state. The *colored piano-hinged loop problem* asks whether a given colored polyomino pattern can be similarly constructed from a given colored piano-hinged loop.

The piano-hinged loop has a simple checkerboarding property:

Observation 1 Consider two adjacent squares u and v in a polyomino P, obtained as a folded state of a piano-hinged loop. Without loss of generality, assume that the top side of u contains (the front side of) triangle p_i. Then (1) the other triangle of u on front side is p_j for some j, (2) the backside of u contains two qs, (3) the front side of v contains two ps, and (4) the backside of v contains two qs.

Ivan's Hinge has a group of triangles that are monochromatic as assumed above, and a group of triangles with different colors on their front and back sides. However, these groups directly correspond to the parity classes in Observation 1. Hence, for each unit square, the front side consists of two triangles from the same group, and the back side consists of two triangles from the other group. Thus, from a theoretical point of view, we can again effectively assume that the pieces are monochromatic. (Practically, the differing colors can vary the color patterns, which can help visually.)

3 Uncolored Piano-Hinged Loop

We begin with the universality theorem of GeoLoop, claimed by Frederickson [4]:

Theorem 1 ([4]) *Any polyomino P of n unit squares can be realized as a folded state of the piano-hinge loop of $4n$ pieces.*

Once we fix the spanning tree T of $G(P)$, we claim that the folded state is uniquely determined up to cyclic shift of the pieces. Both this corollary and the previous theorem follow from a simple argument of repeatedly pruning leaves in the graph of joinings.

Corollary 2 *Let P be any polyomino of n unit squares such that $G(P)$ is a tree. Then it can be uniquely folded from the piano-hinge loop of $4n$ pieces, up to cyclic shift of the pieces.*

For a given tree-shaped polyomino, the piano-hinge loop traverses the tree in the same manner as a depth-first search without crossing. That is, if we imagine that we are in the maze in the form of the tree, and traverse the maze by the right-hand rule, then we traverse each edge twice, and this is the order followed by the piano-hinge loop. This intuition will be useful in some proofs in this paper.

4 General Piano-Hinged Loop

Consider a polyomino P in which pieces p_i and q_i have colors $c(p_i)$ and $c(q_i)$, respectively. When the connection graph $G(P)$ is a tree (or the spanning tree of $G(P)$ is explicitly given), we still have a polynomial-time algorithm to solve the problem:

Theorem 3 *Let P be any polyomino of n unit squares such that $G(P)$ is a tree T. Then the general piano-hinge loop problem can be solved in $O(n^2)$ time.*

Next we turn to the case that P is a general polyomino, where the problem is NP-complete:

Theorem 4 *The colored piano-hinge loop problem is NP-complete, even if the number of colors is 3 and the target polyomino is a rectangle.*

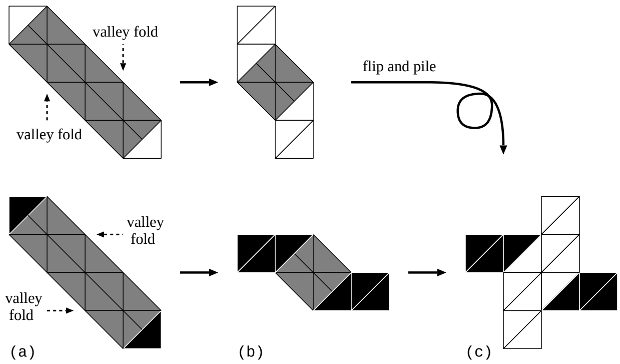

Figure 4: Crossover gadget

Proof outline: We prove NP-hardness by reducing from 3-PARTITION, defined as follows.

3-PARTITION (cf. [5])
INSTANCE: A finite set $A = \{a_1, a_2, \ldots, a_{3m}\}$ of $3m$ weighted elements with $w(a_j) \in \mathbb{Z}_+$, where $w(a_j)$ gives the weight of a_j, and a bound $B \in \mathbb{Z}_+$ such that each a_j satisfies $B/4 < w(a_j) < B/2$ and $\sum_{j=1}^{3m} w(a_j) = mB$.
QUESTION: Can A be partitioned into m disjoint sets $A^{(1)}, A^{(2)}, \ldots, A^{(m)}$ such that $\sum_{a_j \in A^{(i)}} w(a_j) = B$ for $1 \leq i \leq m$?

If A has a solution, then each $A^{(i)}$ must contain exactly three items, because $B/4 < w(a_j) < B/2$, for all i and j.

Figure 3 outlines the construction. Our piano-hinge loop L consists of two parts (Figure 3(a)). The first part is a series of black triangles that form m empty *bins*, such that each bin should be filled by B gray unit squares. The second part is a sequence of gray and white triangles. The ith run of $4a_i$ consecutive grey triangles in the sequence represents the weight of an element a_i for each i. White triangles will be used to place the grey items arbitrary into bins. The key point is that each run of gray triangles must be put into exactly one bin, and the grey triangles cannot be split into two or more different bins. Using the property, we simulate 3-PARTITION. Figure 4 shows one of the gadgets from the full proof.

Adding $3m \times (b + 3) - 1$ black squares in the lower right of Figure 3(b), we can make the desired shape a rectangle of size $15m \times (12m + b + 3)$. □

5 Paper Folding Problem

The *double-covering problem* asks whether a given sheet of paper (of arbitrary shape) can doubly cover a given polyomino. A modification of our proof above shows that this problem is also NP-hard:

Theorem 5 *The doubly covering problem is NP-hard, even when the target polyomino is a rectangle.*

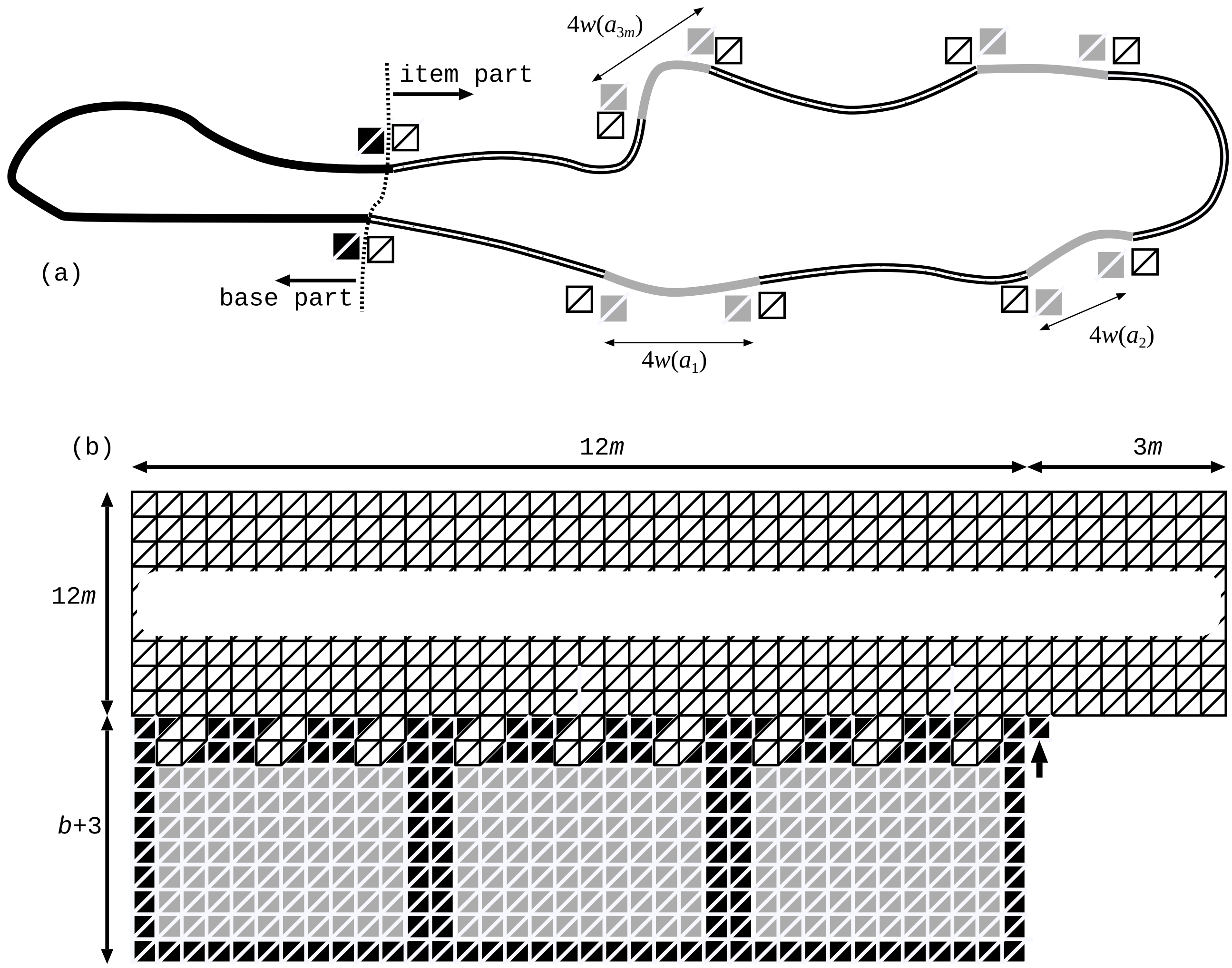

Figure 3: Outline of the construction

Note that this notion of "double covering" a rectangle allows cuts/seams in the middle of the rectangle (along grid edges). Without this flexibility, the folding would be uniquely determined up to cyclic shift, leading to an $O(n^2)$-time algorithm in the same manner as Theorem 3.

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