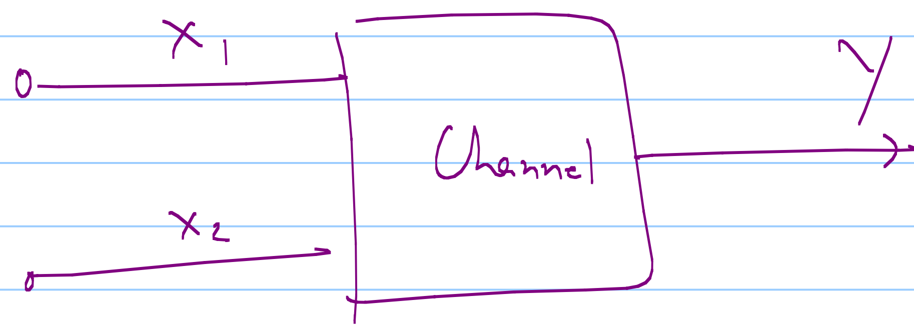
STO 6 LECTURE 20

Note Title 4/27/2006

Today

- Multiple Access Channels (contd.)
- Coding correlated sources (Slepian-Wolf Theorem)

Recall: Mult. Access Channels



Described by Ω_{X_1} Ω_{X_2} Ω_Y

$P_{Y|(X_1, X_2)}$

(R_1, R_2) acheivable if $\exists$ encoding functions

$$X_1 : \{1 \ldots 2^{nR_1}\} \to \Omega_{x_1}^n$$

$$X_2 : \{1 \ldots 2^{nR_2}\} \to \Omega_{x_2}^n$$

$$D : (\Omega_{x_1} \times \Omega_{x_2})^n \to \{1 \ldots 2^{nR_1}\} \times \{1 \ldots 2^{nR_2}\}$$

Crucial

s.t. if W_1, W_2 uniform $\boxed{\text{ind}}$

&

$$X_1(W_1), X_2(W_2) \xrightarrow{\text{Channel}} Y \xrightarrow{D} \hat{W}_1, \hat{W}_2$$

then $P_{err} \triangleq \Pr_{W_1, W_2, Y}\left[\hat{W}_1, \hat{W}_2 \neq W_1, W_2\right] \to 0$

M.A. Coding Theorem

(R_1, R_2) "basic acheivable" if $\exists$ P_{x_1}, P_{x_2}

$$(x_1, x_2) \sim P_{x_1} \times P_{x_2}$$

$$R_1 \leq I(X_1 ; Y \mid X_2)$$

$$R_2 \leq I(X_2 ; Y \mid X_1)$$

$$R_1 + R_2 \leq I((X_1, X_2) ; Y)$$

(R_1, R_2) acheivable iff

$\exists$ basic acheivable $(R_1^{(1)}, R_2^{(1)}) \ldots (R_1^{(k)}, R_2^{(k)})$

& $\lambda_1 \ldots \lambda_k$ where

$$0 \leq \lambda_i \leq 1 \qquad \sum \lambda_i = 1$$

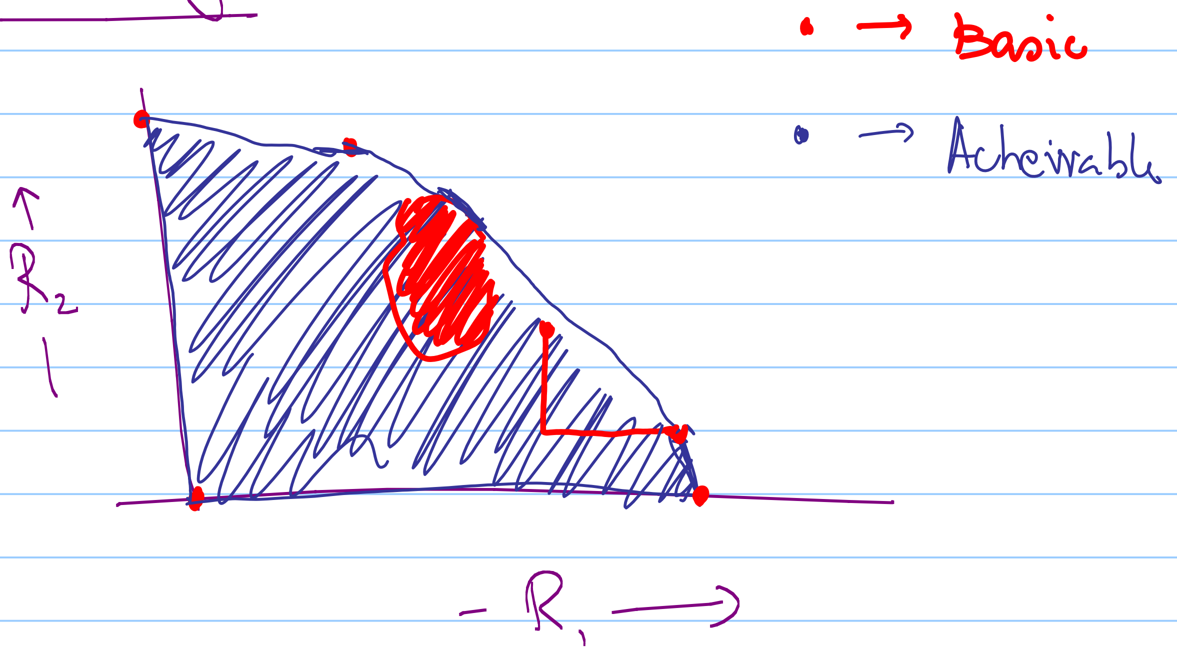
s.t.

$$R_1 = \sum \lambda_i R_1^{(i)}$$

$$R_2 = \sum \lambda_i R_2^{(i)}$$

In other words (R_1, R_2) in <u>convex hull</u> of <u>basic acheivable</u> $(R_1^{(i)}, R_2^{(i)})$.

Pictorially



Proof of achievability

1. Prove Basic by random coding
2. Prove Convex Hull by time - sharing ✓

(2 - straight forward.)

For 1: $X_1(W_1)_i \in \Omega_{X_1}$ i.i.d according to P_{X_1}

$X_2(W_2)_i \in \Omega_{X_2}$ i.i.d. acc to P_{X_2}

Decoding : Given Y

if $\exists$ unique W_1, W_2 s.t.

$(\underline{X}_1(W_1), \underline{X}_2(W_2), \underline{Y})$ jointly typical

then output W_1, W_2 else ERROR

Error in Decoding when transmitting W_1, W_2 if

① $(\underline{X}_1(W_1)\ \underline{X}_2(W_2), \underline{Y})$ not jointly typical

(LLN) say $P_r[①] \to 0$

② $\exists\ W_1' \neq W_1$ s.t.

$(X_1(W_1'), X_2(W_2), \underline{Y})$ jointly typical

$$P_r[②] \leq 2^{R_1 n} \cdot 2^{-I(X_1; X_2, Y) \cdot n}$$

$$= 2^{R_1 n} \cdot 2^{-I(X_1; Y | X_2) \cdot n} \to 0$$

(= by Chain rule & ind. of X_1, X_2).

(3) $\exists\, w_2' \neq w_2$ s.t.

$(X_1(w_1), X_2(w_2'), Y)$ jointly typical

$Pr[(3)] \to 0$ if $R_2 < I(X_2; Y | X_1)$

(4) $\exists\, w_1' \neq w_1$, $w_2' \neq w_2$ s.t.

$(X_1(w_1'), X_2(w_2'), Y)$ jointly typical

$Pr[(4)] \to 0$ if $R_1 + R_2 < I(X_1 X_2; Y)$.

On the converse

How do we deal with "convex hull"?

need to show $\exists$ auxiliary variable Q

(Q = time)

& distribution (X_1, X_2, Y, Q)

s.t. $\forall q$ $(X_1, X_2, Y | Q = q) \sim$ channel

& $R_1 \le I(X_1; Y | X_2, Q)$

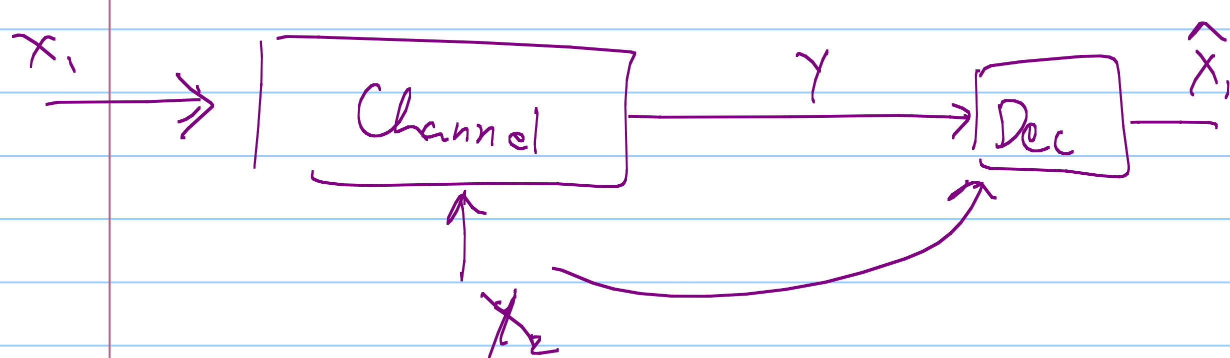
$R_2 \le I(X_2; Y | X_1, Q)$

$R_1 + R_2 \le I(X_1, X_2; Y | Q)$

But will prove this for every $q \in Q$

$$\left. \begin{array}{l} R_{1,q} \le I(X_1; Y | X_2, q) \\ \\ \cdot \end{array} \right\}$$

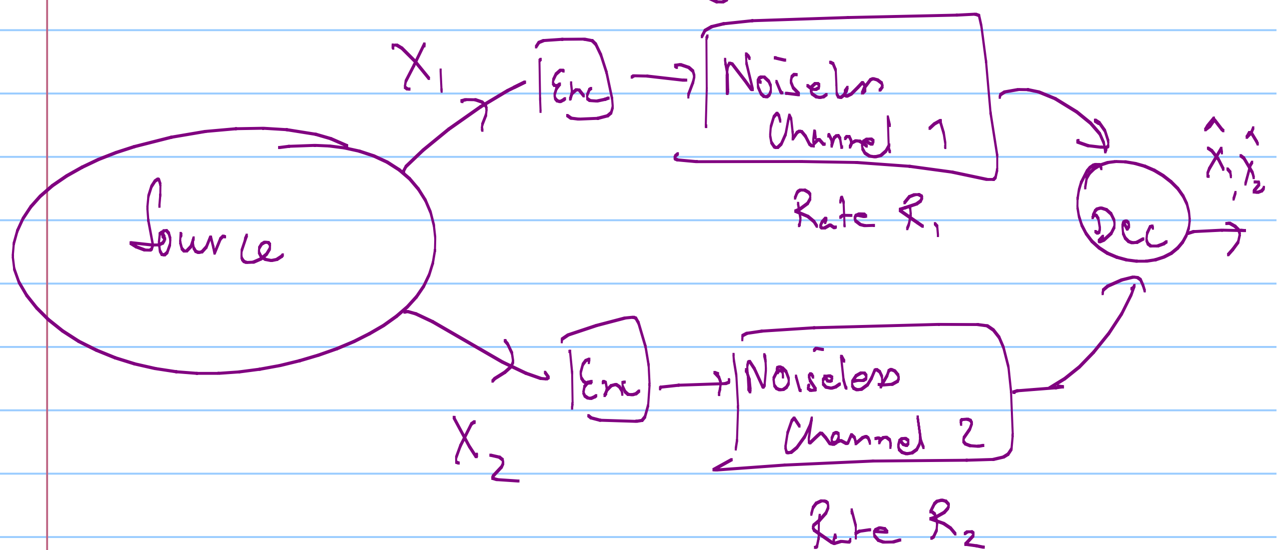
Now this is standard converse.



(X_2 induces "error" in channel, but known to decoder). [See proof in book.]

NEW TOPIC

"Correlated-Source Coding"



(X_1, X_2) not independent

Definition

"Rate (R_1, R_2) suffice if $\exists$

$$E_1 : \Omega_{X_1}^n \longrightarrow \{1 \dots 2^{nR_1}\}$$

$$E_2 : \Omega_{X_2}^n \longrightarrow \{1 \dots 2^{nR_2}\}$$

$$D : \{ \quad \} \times \{ \quad \} \to \Omega_{X_1}^n \times \Omega_{X_2}^n$$

with $P_{err} \to 0$

<u>Obvious</u> $R_1 \geq H(X_1)$

$R_2 \geq H(X_2)$

suffice.

Can we do better?

<u>Examples</u>

$Z_0, Z_1, Z_2, Z_3 \sim \text{Bern}\left(\frac{1}{2}\right)$ i.i.d

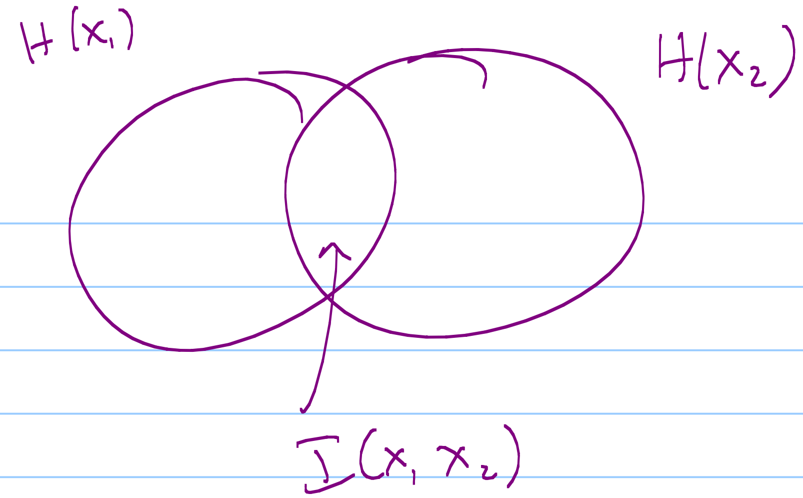
$X_1 = Z_0 Z_1 Z_2$

$X_2 = Z_0 Z_1 Z_3$

$R_1 = 1 \quad R_2 = 3$ ✓

$R_1 = 3 \quad R_2 = 1$ ✓

$R_1 = 2 \quad R_3 = 2$ ✓



Conjecture: $R_1 + R_2 \geq H(X_1, X_2)$ suffices?

$R_1 \geq H(X_1 | X_2)$

$R_2 \geq H(X_2 | X_1)$

Theorem (Slepian-Wolf): True.

<u>Converse: Obvious</u>

<u>Proof of Coding Theorem</u>

Let $H_1 = H(X_1)$

$H_2 = H(X_2)$

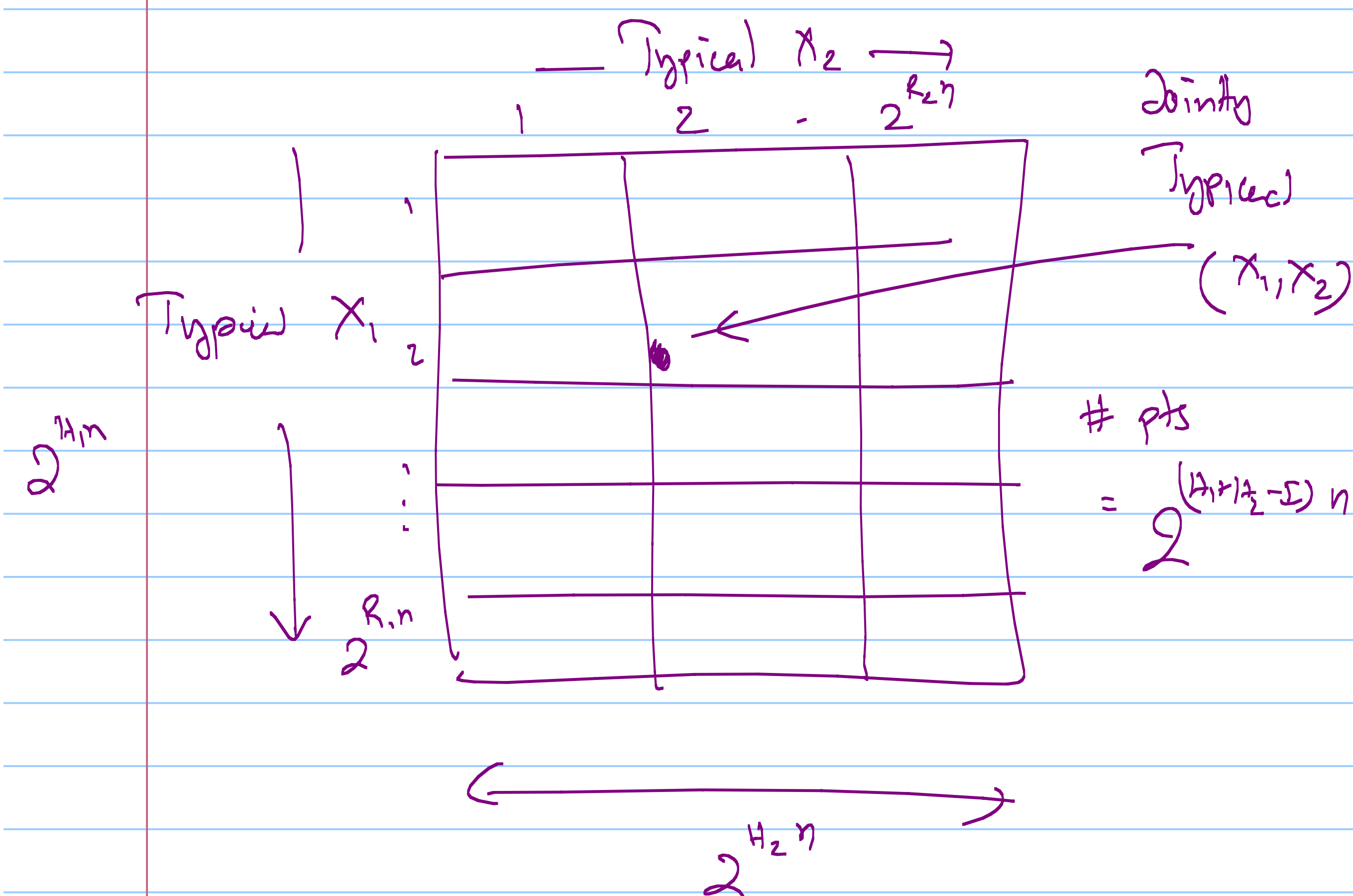
$I = I(X_1 ; X_2)$

(Want to prove $R_1 + R_2 \geq H_1 + H_2 - I$

$R_1 \geq H_1 - I$

$R_2 \geq H_2 - I$

suffices)



Need to describe $(\underline{X}_1, \underline{X}_2)$ by W_1, W_2

$W_1 = f_1(X_1) \qquad W_2 = f_2(X_2)$

Essentially only possible idea (given what we know)

Pick f_1 at random

f_2 at random

Decoding $D^{[w_1, w_2]}$: if $\exists!\,(x_1, x_2)$ jointly typical

s.t. $\big(f_1(x_1), f_2(x_2)\big) = (w_1, w_2)$

then output x_1, x_2

else ERROR

$\Pr[\text{Error}]$:

① (x_1, x_2) not jointly typical $\xrightarrow{\text{Prob}} 0$

② $\exists\ (x_1', x_2') \neq (x_1, x_2)$ s.t.

(x_1', x_2') typical &

$f_1(x_1') = W_1 \quad f_2(x_2') = W_2$

2a $\quad X_1' = X_1$

2b $\quad X_2' = X_2$

2c $\quad$ Neither of the above:

in this case $\Pr\big[f_1(x_1') = W_1$ for fixed typical $x_1' x_2'$

$f_2(x_2') = W_2\big]$

$= 2^{-(R_1+R_2)n}$

$\#$ typical pairs $\leq 2^{(H_1+H_2-I)n}$

$\Rightarrow$ --- $\to 0$

2a $\quad \Pr\big[f_2(x_2') = W_2\big] = 2^{-R_2 n}$

$\#\ X_2'$ s.t. (X_1, X_2') typical $= ?$

$\leq 2^{(H_2-I)n}$

Why?

$$\Pr\left[(x_1', x_2')\right] = 2^{-(H_1+H_2-I)n}$$

$$\Pr\left[x_1'\right] = 2^{-H_1 n}$$

$$\Rightarrow \Pr\left[x_2' \mid x_1'\right] = 2^{-(H_2-I)n}$$

$$\Rightarrow \# x_2' \leq 2^{+(H_2-I)n}$$

$\boxtimes$

Thoughts

- Power of Joint AEP

- Why typical, not most probable?