ST 06 LECTURE 10

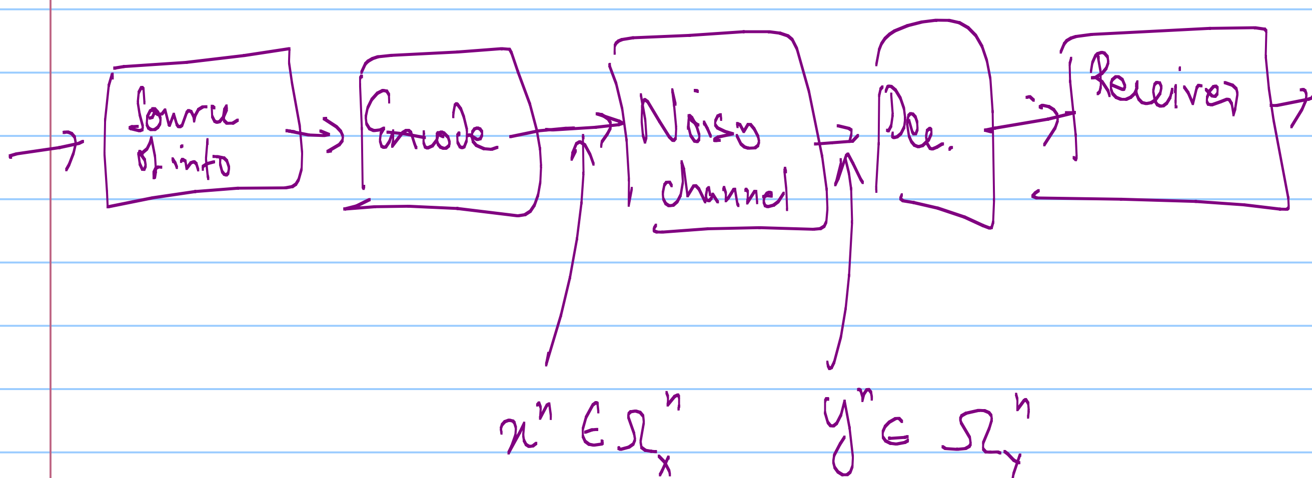
Note Title 3/14/2006

Today:

Channel Coding

- Channel Capacity
- Example Channels
- Joint AEP

Goal



How can we maximize rate at which information is sent?

Given definition / motivation of "information"

Want to maximize $I(X^n; Y^n)$

Where does X^n come from?

Where does Y^n come from?

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(1) X^n : Designer's choice.

Use this to maximize

$$I(X^n; Y^n).$$

Note $I(X^n; Y^n) \le H(X^n)$ ← want this high!

2 Y^n : Induced by

1 Channel Characteristics

& 2 X^n ....

"Simple Class of Channels" a la i.i.d. sources.

- Discrete Memoryless Channels (DMC)

$$P_{Y^n|X^n}(\bar{y}|\bar{x}) = \prod_{i=1}^{n} P_{Y|X}(y_i|x_i)$$

TIME INVARIANT ⟵ independent of i;

INDEPEDENT ⟵ independent of x_j, y_j, $j \neq i$.

Channel Capacity:

$$C^{(n)} = \frac{1}{n} \max_{P_{X^{(n)}}(\bar{x})} I(\bar{X}; \bar{Y})$$

$$\lim_{n\to\infty} C^{(n)} = \text{"Channel Capacity"}$$

Over the course of this lecture & next, will study

- Channel Capacity vs. "n";
- Capacity of simple channels;
- <u>Coding Theorem</u>: As $n \to \infty$ can achieve "reliable" comm. over DMC at "rate < capacity"
- <u>Converse Coding Theorem</u>: Can't do reliable comm. at rate > capacity.

Channel Capacity for DMC

$$I(X^n; Y^n) = H(Y^n) - H(Y^n | X^n)$$

$$= \sum_{i=1}^{n} H(Y_i | Y_1 \ldots Y_{i-1}) - \sum_{i=1}^{n} H(Y_i | X_i)$$

$$\leq \sum_{i=1}^{n} H(Y_i) - \sum H(Y_i | X_i)$$

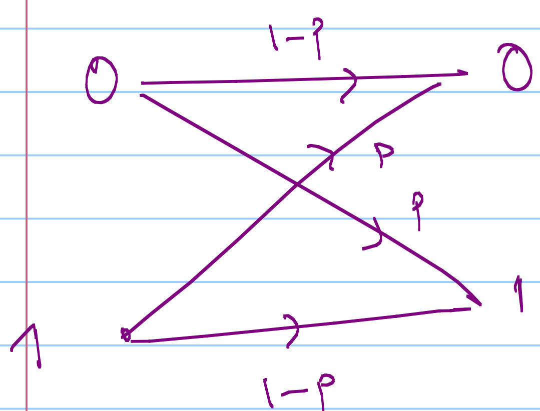
$$\leq n \cdot C^{(1)}$$

But if $X^{(n)}$ = prod. dist over $p(x)$ maximizing $C^{(1)}$ then equality is achieved.

CONCLUDE: $C^{(1)}$ is quantity to be studied!

Common Channels

Binary Symmetric Channel :



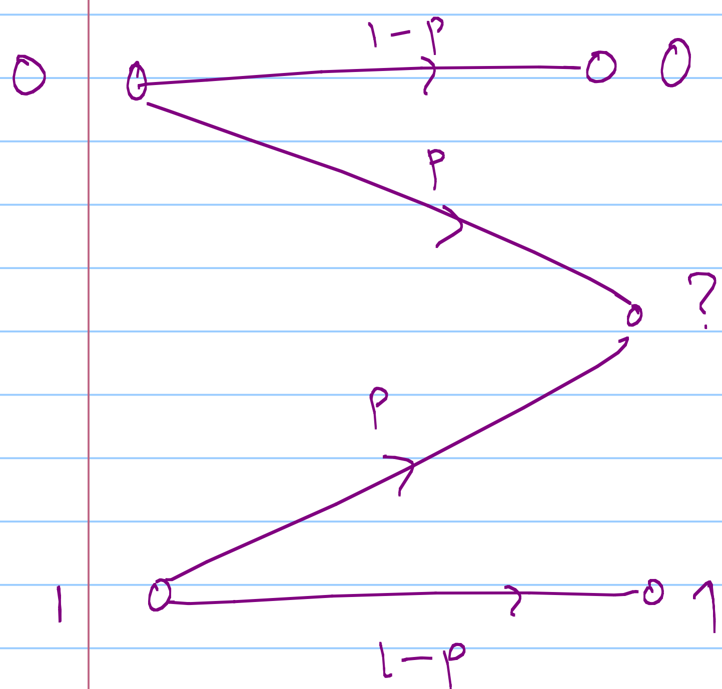
$$C^{(1)} = \max_{p(x)} I(X;Y)$$

$$= \max_{p(\bar{x})} \left\{ H(Y) - H(Y|X) \right\}$$

$$= \max_{p(\bar{x})} \left\{ H(Y) \right\} - H(p)$$

$$\leq 1 - H(p)$$ [Achieved if $X = 0$ w.p. $\frac{1}{2}$, $= 1$ w.p. $\frac{1}{2}$]

BINARY ERASURE CHANNEL

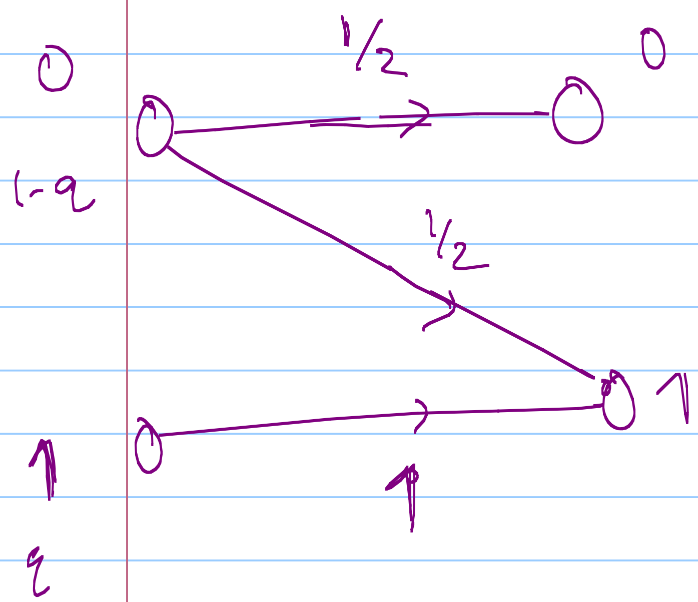


$$C = \max_{p(x)} \left\{ H(X) - H(X \mid Y) \right\}$$

$$H(X \mid Y) = 0 \cdot P_r\left[Y \neq ? \right] + H(X) \cdot P_r\left[Y = ? \right] = p \cdot H(X)$$

$$C = H(X)(1-p) \leq 1-p$$ [Achieved if $H(X) = 1$

Example Channel Where Non-Uniform $P(x)$ is better



$$C = \max_{q} \left\{ H(q) - \left(1 - \frac{q}{2}\right) \cdot H\left(\frac{q/2}{1 - q/2}\right) \right\}$$

Optimized at $q \neq \frac{1}{2}$ . . .

Symmetric Channels

Channel Symmetric if all rows are permutation of each other AND all columns are permutations of each other.

Property of Symmetric Channel

Uniform dist. on X induces uniform distribn on Y.

$$\text{Cap} = \max_{p(x)} \{ H(Y) - H(Y|X) \}$$

$$= \log |Y| - H(\text{row})$$

E.g.

$$26 \updownarrow \overbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ & & \vdots & & & \end{bmatrix}}^{26}$$

"Noisy typewriter"

AEP for Channel Coding

Channel maps $\bar{X}^n \longrightarrow \bar{Y}^n$

then (x^n, y^n) is "ϵ-jointly typical" if

① $\left| -\frac{1}{n} \log P_X(x^n) - H(X) \right| < \epsilon$

AND ② $\left| -\frac{1}{n} \log P_Y(y^n) - H(Y) \right| < \epsilon$

AND ③ $\left| -\frac{1}{n} \log P_{X^n,Y^n}(x^n, y^n) - H(X,Y) \right| < \epsilon$

AEP Theorem

- $\left| A_\epsilon^{(n)} \right| \leq 2^{(\epsilon + H(X,Y)) \cdot n}$
- $P\left(\bar{X}^n, \bar{Y}^n\right) \in A_\epsilon^{(n)} \longrightarrow 1$ as $n \to \infty$.