Unfolding and Dissection of Multiple Cubes

Zach Abel[1] Brad Ballinger[2] Erik D. Demaine[1] Martin L. Demaine[1]
Jeff Erickson[3] Adam Hesterberg[1] Hiro Ito[4] Irina Kostitsyna[5]
Jayson Lynch[1] Ryuhei Uehara[6]

A *polyomino* is a "simply connected" set of unit squares introduced by Solomon W. Golomb in 1954. Since then, a set of polyominoes has been playing an important role in puzzle society (see, e.g., [3, 1]). In Figure 82 in [1], it is shown that a set of 12 pentominoes exactly covers a cube that is the square root of 10 units on the side. In 1962, Golomn also proposed an interesting notion called "rep-tiles": a polygon is a rep-tile of order k if it can be divided into k replicas congruent to one another and similar to the original (see [2, Chap 19]).

These notions lead us to the following natural question: is there any polyomino that can be folded to a cube and divided into k polyominoes such that each of them can be folded to a (smaller) cube for some k? That is, a polyomino is a *rep-cube* of order k if itself is a net of a cube, and it can be divided into k polyominoes such that each of them can be folded to a cube. If each of these k polyominoes has the same size, we call the original polyomino *regular* rep-cube of order k. In this paper, we give an affirmative answer. We first give some regular rep-cubes of order k for some specific k. Based on this idea, we give a constructive proof for a series of regular rep-cubes of order $36gk'^2$ for any positive integer k' and an integer g in $\{2, 4, 5, 8, 10, 50\}$. That is, there are infinitely many k that allow regular rep-cube of order k. We also give some non-regular rep-cubes and its variants.

We first show some specific solutions.

Theorem 1 *There exists a regular rep-cube of order k for $k = 2, 4, 5, 8, 9, 36, 50, 64$*

Proof. For each of $k = 2, 4, 5, 8, 9$, we give a regular rep-cube in Figure 1. It is not difficult to see that they satisfy the condition of rep-cubes.

For $k = 36$, we use six copies of the pattern given in Figure 2. Using this pattern, we can combine them into any one of eleven nets of a cube.

For $k = 64$, we use one copy of the left pattern in Figure 2 for the bottom of a big cube, four copies of the center pattern in Figure 2, and one copy of the right pattern in Figure 2 for the top of the big cube. The consistency can be easily observed.

For $k = 50$, we make a program for finding packings of nets of unit cubes on twisted grids on bigger cubes by exhaustive search. We found a packing on a $(7, 1)$ twist, i.e., a dissection of the surface of a $\sqrt{50} \times \sqrt{50} \times \sqrt{50}$ cube into 50 nets of unit cubes as shown in Figure 2. ∎

Based on the solution for $k = 36$ in Theorem 1, we obtain the following theorem:

Theorem 2 *There exists a regular rep-cube of order $36gk'^2$ for any positive integer k' and an integer g in $\{2, 4, 5, 8, 10, 50\}$. That is, there exists an infinite number of regular rep-cubes.*

Proof.(Outline) In each pattern in the proof of Theorem 1, we first split each unit square into k'^2 small squares. Then we replace each of them by the pattern for $k = 36$ in Figure 2, and obtain the theorem. ∎

One may think that non-regular rep-cubes are more difficult than regular ones. So far, we have found some:

Theorem 3 *There exists a non-regular rep-cube of order k for $k = 2, 10$.*

Proof. For $k = 2$, the rep-cube is given in Figure 3(left): this itself folds to a cube of size $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$, and it can be cut into two pieces such that one folds into a cube of size $2 \times 2 \times 2$, and the other folds into a unit cube. We note that these areas satisfy $6 \times (\sqrt{5})^2 = 6 \times 1^2 + 6 \times 2^2 = 30$.

For $k = 10$, the rep-cube is given in Figure 3(right): this pattern contains 150 unit squares. It is easy to see that nice nets of unit cube use 54 unit squares in total. The remaining 96 squares form a net of cube of size $4 \times 4 \times 4$. Moreover, this pattern also folds to a cube of size $5 \times 5 \times 5$. These areas satisfy $150 = 6 \times 5^2 = 6 \times (3^2) + 6 \times (4^2) = 6(3^2 + 4^2)$. ∎

In this paper, we introduce a new notion of "rep-cube," and show several examples. So far, we have no systematic ways to investigate them. However, from the trivial constraint for the areas, we can consider many variants as shown in the last example for $k = 10$: Is there a rep-cube of order 6 from a $3 \times 3 \times 3$ cube into

[1] MIT, {zabel, edemaine, mdemaine, achester, jaysonl}@mit.edu
[2] Humboldt State Univ., ballingerb@gmail.com
[3] UIUC, jeffe@cs.uiuc.edu
[4] UEC, itohiro@uec.ac.jp
[5] ULB, irina.kostitsyna@ulb.ac.be
[6] JAIST, uehara@jaist.ac.jp

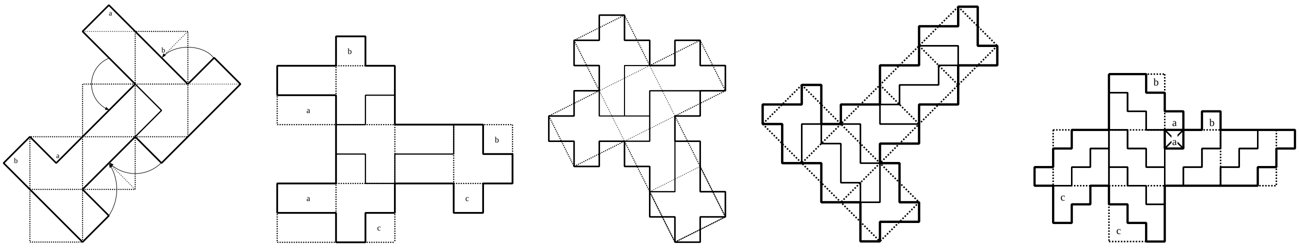

Figure 1: Rep-cubes of order $k = 2, 4, 5, 8, 9$, respectively.

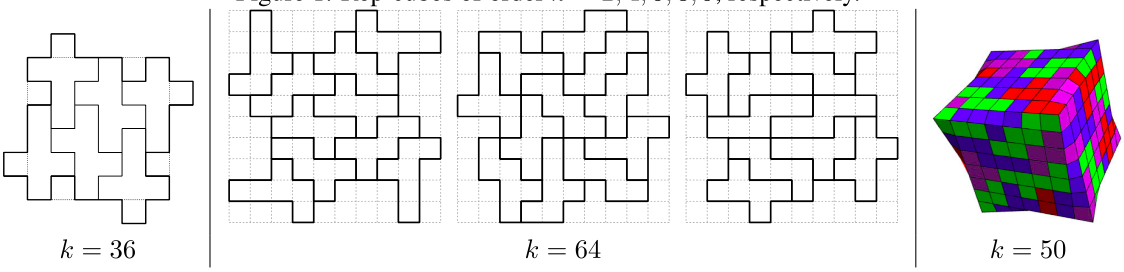

Figure 2: Patterns for rep-cubes of order $k = 36$, $k = 64$, and $k = 50$

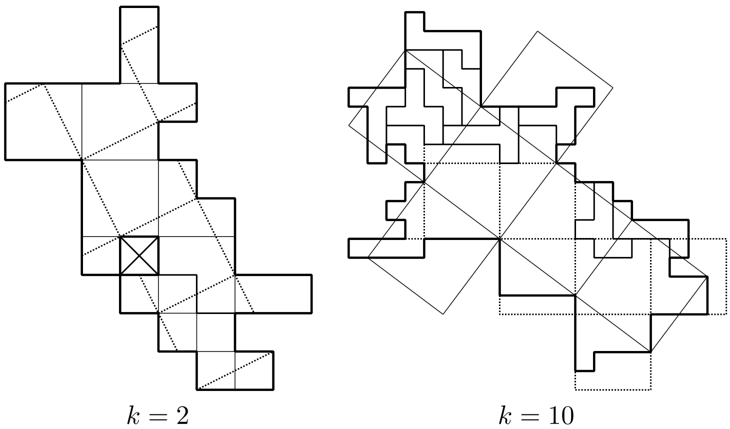

Figure 3: Patterns for non-regular rep-cubes of order $k = 2, 10$.

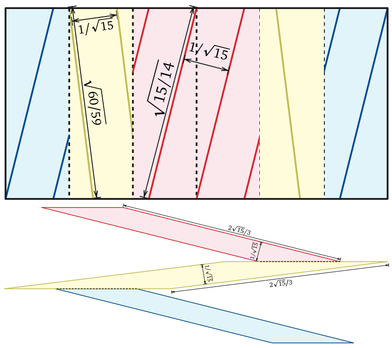

Figure 4: One doubly covered square to three doubly covered squares.

one $2 \times 2 \times 2$ cube and five $1 \times 1 \times 1$ cubes, and so on. Especially, one interesting open question is that is there are rep-cube of order 2 from one $5 \times 5 \times 5$ cube into one $4 \times 4 \times 4$ cube and $3 \times 3 \times 3$ cube. We note that this size comes from the Pythagoras triangle $3^2 + 4^2 = 5^2$. We have already known that there are infinitely many Pythagoras triangles. For each of them, can we construct a rep-cube of order 2?

Is there any integer k such that we have no regular rep-cube of order k? It seems to be unlikely that there is a regular rep-cube of order 3. How can we prove that? In this paper, we also introduce "regular" rep-cubes. One natural additional condition may be making every small development congruent; for example, each example for $k = 2, 4, 9$ satisfies this condition. What happens if we employ this additional condition?

One of other extensions is different dimension and shape. For example, we have the following theorem:

Theorem 4 *Let $A, a_1, \ldots, a_k$ be any positive real numbers such that $\sum_i a_i = A$. (1) There is a net of a doubly-covered square with area A that can be cut into k polygons with areas $a_1, \ldots, a_k$, each of which can be folded into a double-covered square (see Figure 4 for $k = 3$). (2) There is a net of a regular tetrahedron with area A that can be cut into k polygons with areas $a_1, \ldots, a_k$, each of which can be folded into a regular tetrahedron.*

References

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