Deflating The Pentagon

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Abstract. In this paper we consider deflations (inverse pocket flips) of pentagons. We show that every pentagon can be deflated after finitely many deflations, and that any infinite deflation sequence of a pentagon results from deflating an induced quadrilateral on four of the vertices.

1 Introduction. A *deflation* of a simple planar polygon is the operation of reflecting a subchain of the polygon through the line connecting its endpoints such that (1) the line intersects the polygon only at those two polygon vertices, (2) the resulting polygon is simple (does not self-intersect), and (3) the reflected subchain lies inside the hull of the resulting polygon. A polygon is *deflated* if it does not admit any deflations, i.e., every pair of polygon vertices either defines a line intersecting the polygon elsewhere or results in a nonsimple polygon after reflection.

Deflation is the inverse operation of pocket flipping. Given a nonconvex simple planar polygon, a *pocket* is a maximal connected region exterior to the polygon and interior to its convex hull. Such a pocket is bounded by one edge of the convex hull of the polygon, called the *pocket lid*, and a subchain of the polygon, called the *pocket subchain*. A *pocket flip* (or simply *flip*) is the operation of reflecting the pocket subchain through the line extending the pocket lid. The result is a new, simple polygon of larger area with the same edge lengths as the original polygon. A convex polygon has no pocket and hence does not admit a flip.

In 1935, Erdős conjectured that every nonconvex polygon convexifies after a finite number of flips [3]. Nagy [5] claimed a proof of Erdős's conjecture. Recently, Demaine et al. [2] uncovered a flaw in Nagy's argument, as well as other claimed proofs, but fortunately correct proofs remain.

In the same spirit of finite flips, Wegner conjectured in 1993 that any polygon becomes deflated after a finite number of deflations [7]. Eight years later, Fevens et al. [4] disproved Wegner's conjecture by demonstrating a family of quadrilaterals that admit an infinite number of deflations. They left an open problem of characterizing which polygons deflate infinitely. Ballinger [1] closed the problem for quadrilaterals by proving that all infinitely deflating quadrilaterals are in the family of Fevens et al.

In this paper, we study deflations of pentagons. We prove that every pentagon admitting an infinite number of deflations induces an infinitely deflatable quadrilateral on four of its vertices. Then we show our main result: unlike quadrilaterals, every pentagon can be deflated after finitely many (well-chosen) deflations.

2 Definitions and Notations. Let $P = \langle v_0, v_1, \ldots, v_{n-1} \rangle$ be a polygon together with a clockwise ordering of its vertices. Let $P^k = \langle v_0^k, v_1^k, \ldots, v_{n-1}^k \rangle$ denote the polygon after k arbitrary deflations, and P^* denote the limit of P^k, when it exists, having vertices v_i^*. Thus, the initial polygon $P = P^0$. A *flat polygon* is a polygon with all its vertices collinear. A *hairpin* vertex v_i is a vertex whose incident edges overlap each other, i.e., forming a turn angle of 180°.

3 Deflation in General. In this section, we describe general properties about deflation of arbitrary simple polygons. Our first few lemmata are fairly straightforward, while the last lemma is quite intricate and central to our later arguments.

Lemma 1 *In any infinite deflation sequence $P^0, P^1, P^2, \ldots$, v_i^* is a hairpin vertex in some accumulation point P^* if and only if v_i^* is a hairpin vertex in all accumulation points P^*.*

Lemma 2 *Any n-gon with n odd and no flat vertices cannot flatten in an accumulation point of an infinite deflation sequence.*

Lemma 3 *For any infinite deflation sequence $P^0, P^1, P^2, \ldots$, there is a subchain $v_i, v_{i+1}, \ldots, v_j$ (where $j - i \geq 2$) that is the pocket chain of infinitely many deflations.*

Lemma 4 *Assume $P = P^0$ has no flat vertices. If P^* is an accumulation point of the infinite deflation sequence $P^0, P^1, P^2, \ldots$, and subchain $v_i, v_{i+1}, \ldots, v_j$ (where $j - i \geq 2$) is the pocket chain of infinitely many deflations, then $v_i^*, v_{i+1}^*, \ldots, v_j^*$ are collinear and $v_{i+1}^*, \ldots, v_{j-1}^*$ are hairpin vertices. Furthermore, if $v_{i+1}^*, \ldots, v_{j-1}^*$ extends beyond v_j^*, then v_j^* is a hairpin vertex; and if $v_{i+1}^*, \ldots, v_{j-1}^*$ extends beyond v_i^*, then v_i^* is a hairpin vertex. In particular, if $j - i = 2$, then either v_i^* or v_j^* is a hairpin vertex.*

4 Deflating Quadrilaterals. We briefly review facts about quadrilateral deflation proved by Fevens et al. [4] and Ballinger [1].

Lemma 5 [1] *Any accumulation point of an infinite deflation sequence of a quadrilateral is flat and has no flat vertices.*

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