

Common Unfoldings of Polyominoes and Polycubes

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Abstract. This paper studies common unfoldings of various classes of polycubes, as well as a new type of unfolding of polyominoes. Previously, Knuth and Miller found a common unfolding of all tree-like tetracubes. By contrast, we show here that all 23 tree-like pentacubes have no such common unfolding, although 22 of them have a common unfolding. On the positive side, we show that there is an unfolding common to all “non-spiraling” k -ominoes, a result that extends to planar non-spiraling k -cubes.

1 Introduction

POLYOMINOES. A *polyomino* or *k-omino* is a weakly simple orthogonal polygon made from k unit squares placed on a unit square grid. Here the polygon explicitly specifies the *boundary*, so that two squares might share two grid points yet we consider them to be not adjacent. Two unit squares of the polyomino are *adjacent* if their shared side is interior to the polygon; otherwise, the intervening two edges of polygon boundary are called an *incision*. By imagining the boundary as a cycle (essentially an Euler tour), each edge of the boundary has a unique incident unit square of the polyomino.

In this paper, we consider only “tree-like” polyominoes. A polyomino is *tree-like* if its dual graph, which has a vertex for each unit square and an edge for each adjacency (as defined above), is a tree. See Figure 1. By double-counting overlapping boundary segments, every tree-like k -omino has a boundary consisting of $2k+2$ unit edges. A subclass of tree-like polyominoes is the class of *path-like* polyominoes, whose dual graph is a path.

POLYCUBES. The 3D analogs of polyominoes are *polycubes*—weakly simple orthogonal polyhedra made from unit cubes on the unit cube grid. As before, two cubes are *adjacent* if their shared side is interior to the polyhedron, and

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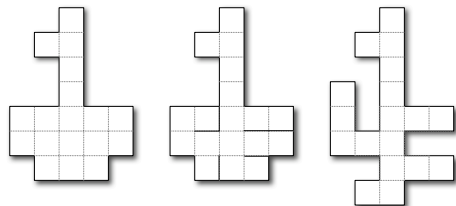


Fig. 1. Three octakaidecominoes (18-ominoes): non-tree-like (left), tree-like (middle), and incision-free tree-like (right).

otherwise the intervening polyhedron surface is an *incision*. The *surface* of the polycube consists of square sides of cubes that have no adjacent cube along that side (either from empty space or from an incision).

In this paper, we consider only *tree-like* polycubes, where the dual graph is a tree. The surface area of every tree-like k -cube is $4k+2$.

In a *planar* polycube, the bottoms of all cubes lie in the xy plane. Thus, planar polycubes are only one cube thick.

A polycube and its mirror image are considered *distinct* if there is no rigid motion that transforms one onto the other. Such pairs of polycubes are called *chiral twins*.

UNFOLDING. For the purposes of this paper, an *unfolding* of a polycube involves cutting a collection of edges of the polycube's surface so that the resulting manifold with boundary can be placed as a polyomino in the plane. One may think of the polycube faces as remaining rigid, with their common edges serving as flexible creases. Note that we are not concerned with the motion that makes the manifold planar. It is only the final planar non-overlapping state that matters.

In the literature on unfolding orthogonal polyhedra [2, 4–8, 10, 14], such an unfolding is usually called a (weakly simple) *grid unfolding*, because it only cuts along grid lines.

We also define a related notion of unfolding polyominoes into more linear structures, though our notion borrows some of the spirit of the vertex-grid unfoldings from [9, 6, 10].

A *subpolyomino* is a connected subset of squares from a polyomino P , containing one designated edge from the boundary of P , called a *flap*. A *cut* subpolyomino is obtained by separating its interior from its entire boundary, except for its flap. A *partition* of P is a collection of interior-disjoint cut subpolyominoes whose closure of their union is P .

The boundary of a given polyomino P can be cut at a vertex, to produce a (weakly simple) chain. To begin an *unfolding* of P we partition P and cut its boundary at a vertex. Then we straighten the boundary chain by straightening the angle at every vertex, via rotation. Meanwhile we maintain the connection between every cut sub-polyomino with its flap (see Figure 2). The motion of this unfolding does not concern us. What matters is the mapping from P to its

unfolded state. If no two squares map to the same location, the unfolding is *valid*.

By construction, every subpolyomino lies on the same side of the straightened boundary (corresponding to the interior). Every polyomino has a trivial valid unfolding by partitioning into just one subpolyomino, leaving it connected to an arbitrary flap, and cutting all other boundary edges.

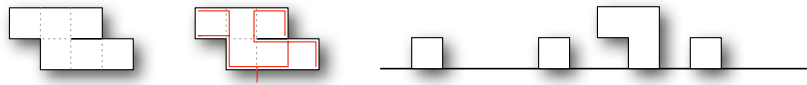


Fig. 2. A possible unfolding of a tree-like 6-omino.

COMMON UNFOLDINGS. Let G be a collection of polycubes (or polyominoes). If each member of G can unfold to the same shape, then this shape is a *common unfolding* of G . The central questions addressed in this paper are (1) do all tree-like k -ominoes have a common unfolding? and (2) do all tree-like k -cubes have a common unfolding?

When unfolding polycubes, one subtlety is whether the common unfolding has an orientation, specifying which side of the polyomino is on top. A common unfolding H is *one-sided* if all polycubes in G map their interior surface to the same side of H ; otherwise H is *two-sided*. If G consists of a pair of chiral twins, and if one element of G has an unfolding H , then H is a two-sided (but not necessarily one-sided) common unfolding for G .

RELATED WORK. Knuth and Miller found a common one-sided unfolding of all seven tree-like tetracubes, by an exhaustive search. They used this to produce *Cubigami7* [12], a commercialized puzzle (see Figure 3).

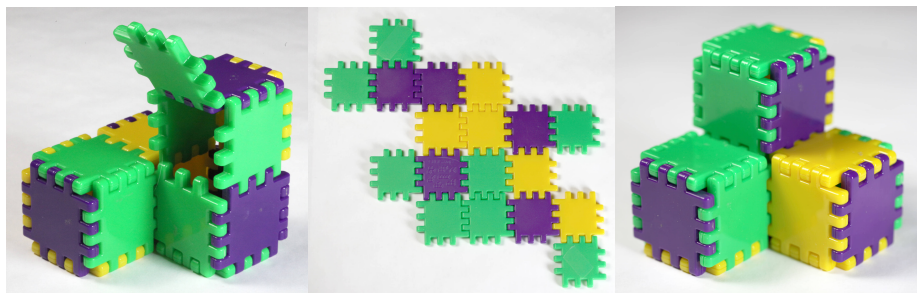


Fig. 3. Cubigami7

Biedl et al. [1] found two examples of a common polyomino unfolding of two distinct boxes (convex polycubes), described in [11, sec. 25.8.3]. Mitani and Uehara [13] have shown that there is an infinite number of such polyominoes. It remains open whether there is a polyomino that folds into more than two distinct boxes.

The survey of O'Rourke [14] covers many results about unfolding orthogonal polyhedra, which include polycubes. An *orthostack* is an orthogonal polyhedron for which every horizontal planar slice is connected. Biedl et al. [2] showed that orthostacks can be unfolded, but require cuts at half-integer coordinates. It remains open whether orthostacks have polyomino (grid) unfoldings. They also showed that *orthotubes* (what we call path-like polycubes) can always be grid unfolded to polyominoes. Damian et al. [4] showed that a subclass of tree-like polycubes called *well-separated orthotrees* can be grid unfolded into polyominoes.

RESULTS. Define $\langle c_1, c_2, \dots, c_{2k+2} \rangle$ to be the ordered sequence of unit edges composing the unfolded boundary of a k -omino where c_1 is the leftmost segment. A *caterpillar* unfolding of a k -omino has one square attached to each edge $c_{2i+\lfloor 2i/(k+1) \rfloor}$ for $1 \leq i \leq k$ (see Figure 4).

The caterpillar unfolding, or just *caterpillar*, cannot fold into all path-like polyominoes. For instance, take a path-like polyomino Q with no two bends adjacent, and k even. Call c_{k+1} and c_{k+2} the *middle edges* of the caterpillar, i.e., the two consecutive edges without flaps. If neither of these two edges map to an edge of an end-square of Q , then we claim that the folded caterpillar will not cover all squares of Q : it suffices to consider four simple local cases to verify that the squares mapped to by the middle edges could only be covered if there existed two consecutive flaps in the unfolding, which does not occur on a caterpillar. Therefore, the middle edges of a caterpillar must map to an end-square of Q . Actually, the same must hold for edges c_1 and c_{2k+2} (i.e., the first and last edges) of the caterpillar. Now, if Q is made to be a spiral with a sufficient number of bends, the length of the boundary on each side between end-squares will differ significantly. Given the symmetry of the caterpillar and the necessary placement of the middle and ends of the caterpillar on the end-squares of Q , this produces a contradiction.

On the other hand, we prove that the caterpillar is a common unfolding of many polyominoes, called “non-spiraling”. Roughly speaking, non-spiraling polyominoes have almost equal boundary lengths between their endpoints. More formally, a *non-spiraling* polyomino is path-like (allowing adjacent bends), but with the additional restriction that the difference between the number of left and right 90° turns does not exceed 3 if k is even, and does not exceed 4 if k is odd. In Section 2 we show that the caterpillar can fold into any *non-spiraling* polyomino. In Section 3.1 we prove a similar result for unfolding non-spiraling planar polycubes into *scolopendra polyominoes* (illustrated in Figure 6).

In Section 3.2 we prove that there exist common one-sided unfoldings of all tree-like incision-free planar k -cubes for $k \leq 6$. In Section 3.3 we show that there is no common unfolding for all tree-like pentacubes.

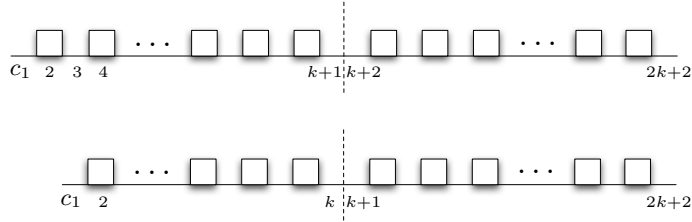


Fig. 4. Caterpillar unfolding of k -ominoes for k even (top) or odd (bottom).

2 Unfolding Path-Like Polyominoes

As mentioned, caterpillars cannot fold into certain spirals. We conjecture that any unfolding of a spiral will not also unfold from a straight path. See Fig. 5.

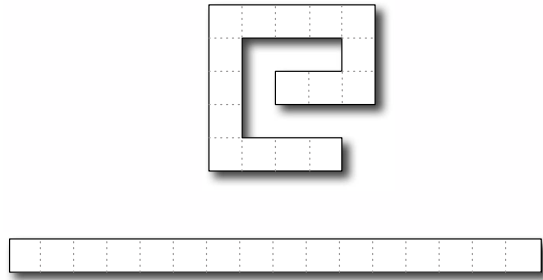


Fig. 5. Any common unfolding of these polyominoes must not be a caterpillar. We conjecture that there is no common unfolding.

Theorem 1. *For any positive value of k , the caterpillar is a common unfolding of all non-spiraling k -ominoes.*

Proof. It suffices to describe how to fold the caterpillar back to any non-spiraling P . Let $\{s_1, s_2, \dots, s_k\}$ be the ordered sequence of unit squares composing P . Let a *non-shared side* be a boundary edge of a square. The squares s_1 and s_k each have three non-shared sides. All squares s_i for $1 < i < k$ have exactly two non-shared sides. A square s_i is a *connector* if its two non-shared sides are opposite, otherwise s_i is a *turn*. We assume without loss of generality that the number of left turns is larger than the number of right turns (otherwise the order of the sequence of squares is simply inverted).

Consider the subpath $P' = \{s_2, \dots, s_{k-1}\}$ of P . The sides of P' form the left subchain L and the right subchain R with $|L| + |R| = 2k - 4$. The length of L or R corresponds to the number of connectors on the path plus twice the number of

right or left turns, respectively. Since P is non-spiraling, $|R| - |L| \leq 6$ for even values of k and $|R| - |L| \leq 8$ for odd.

Suppose that k is even. Let C be the boundary of P . As a first folding step, we match the segments c_{k-1}, c_k and c_{k+1} of C to the sides of s_1 . In general, whenever a single segment c_i is matched to a side s_j , the remainder of the mapping is fully determined (and thus so is the outcome of the folding attempt). Therefore, the left portion of C , $\langle c_1, \dots, c_{k-2} \rangle$ is matched to L , and the right portion $\langle c_{k+2}, \dots, c_{2k+2} \rangle$ is matched to R .

We claim that the two portions of C meet at the end of P , i.e., at a vertex of s_k : After the first step of the folding, the unmatched left and right portions of C have size $k-2$ and $k+1$, respectively. We know that $|R| - 6 \leq |L| \leq |R|$ and $|L| + |R| = 2k-4$ thus $k-5 \leq |L| \leq k-2$ and $k-2 \leq |R| \leq k+1$. Hence the folding of the left and right portions of C is ensured to reach the end of P and there is no overlap at s_k .

This last fact implies that the two boundary edges of every turn belong to the same portion of C , whereas the two boundary edges of every connector belong to a different portion of C . Since there is no more than one square *attached* to every pair of consecutive segments of C (i.e. every such pair includes at most one flap), it follows that a turn does not create an overlap. Note that the first $\lfloor k/2 \rfloor$ squares (on the left portion) of C are attached to even segments and the remaining squares (on the right portion) are attached to odd segments. Thus if the segments of C that match the sides of a connector are either both even or both odd, there is no overlap. We prove this below.

Suppose that we iteratively match segments from both portions of C to squares along P , starting from s_1 . Assume as an induction hypothesis that during the folding process, after the segments of C have been matched to the sides of s_ℓ , both of the first segments of the remaining unmatched left and right portions of C are even, or both are odd. Observe that this is the case for the first step after we match the sides of s_1 . Now if $s_{\ell+1}$ is a connector, the first segment of each unmatched portion of C is matched to the side of $s_{\ell+1}$. Since these two segments are either both even or both odd there is no overlap and the induction hypothesis is satisfied. If $s_{\ell+1}$ is a turn then two segments of one of the unmatched sides of C are matched to the side of $s_{\ell+1}$ and the induction hypothesis is satisfied. This concludes the proof for the even case.

For odd values of k , we have $k-6 \leq |L| \leq k-2$ and $k-2 \leq |R| \leq k+2$. If $|R| \leq k-1$ then the segments c_{k+1}, c_{k+2} and c_{k+3} of C are folded so that they match the sides of s_1 . Otherwise this is done with the segments c_{k-2}, c_{k-1} and c_k . Using similar arguments as in the previous case (k even) we can show that this yields a valid folding. \square

In the following we give a bound on the number of non-spiraling k -ominoes.

Corollary 1. *For any odd k , there exists a common unfolding of a set of at least $\frac{1}{4}C_{(k-1)/2}$ k -ominoes, where C_n is the n th Catalan number.*

Proof. A *Dyck path* [3] of order n is a monotone path along the edges of a $n \times n$ square grid, going from $(0, 0)$ to (n, n) while always remaining below (possibly touching) the diagonal $x = y$. The number of Dyck paths of order n is C_n .

The number of grid vertices on a Dyck path of order n is $2n+1$ and the difference between the number of left and right turns does not differ by more than 1. Thus a Dyck path of order $(k-1)/2$ is the dual of a non-spiraling k -omino. Let D be a given Dyck path. D has at most one other Dyck path as a mirror image (about the $x = -y$ line). Also, D is a 180° -rotation of at most one other Dyck path. Thus there are at least $\frac{1}{4}C_{(k-1)/2}$ distinct Dyck paths (under rotation and reflection). By Theorem 1 the set of polyominoes dual to these distinct Dyck paths have a common caterpillar unfolding. \square

3 Unfolding Polycubes

3.1 Path-Like Planar Polycubes

Every path-like planar polycube has a planar dual, just as any path-like polyomino. Thus *non-spiraling polycubes* are a subclass of path-like planar polycubes, and are defined analogous to non-spiraling polyominoes.

The *scolopendra* is designed to be able to fold to any non-spiraling k -cube. It has a primary structure formed by a row of $2k+2$ unit squares $\{s_1, \dots, s_{2k+2}\}$. Two additional squares, called *feet*, are joined to the remaining sides of each square $s_{2i+[2i/k+1]}$ for $1 < i \leq k$. (see Fig. 6).

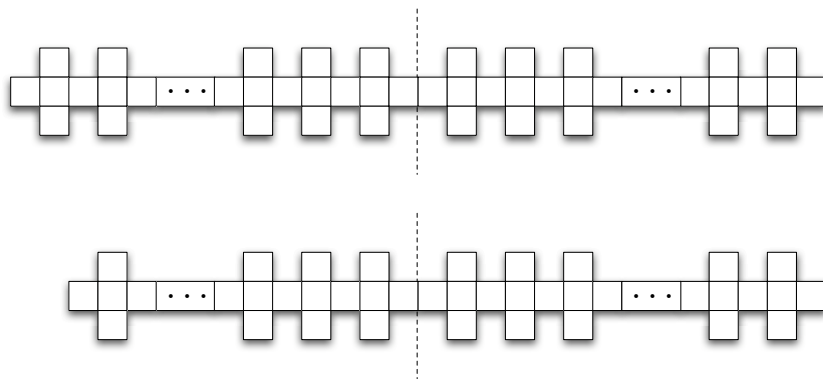


Fig. 6. The scolopendra, for k even and odd.

Theorem 2. *The scolopendra is a common unfolding of all non-spiraling k -cubes.*

Proof. It suffices to show how to fold a scolopendra H to any given k -cube. Fold every foot of H so that it forms an angle of $\pi/2$ with its adjacent square.

Now, the projection of H along the common normal direction of each foot is identical to the caterpillar unfolding shown in Fig. 4. The primary structure of H projects to a line segment. Also, every non-spiraling k -cube projects to its own unique non-spiraling k -omino. By Theorem 1 the projection of H will cover the projection of the k -cube. What this means is that the feet of H will correctly fold to cover all top and bottom faces of the k -cube. Meanwhile, the primary structure of H covers all the side faces of the k -cube, just as the chain of a caterpillar folds to create the boundary of a k -omino in Theorem 1. Thus the scolopendra is a valid unfolding of the given k -cube. \square

Corollary 2. *For any k , there exists a polyomino that folds to at least $\frac{1}{4}C_{(k-1)/2}$ k -cubes, where C_n is the n th Catalan number.*

Proof. Similar to the proof of Corollary 1. \square

3.2 Tree-Like Planar Polycubes

Consider a k -tube, that is, a $1 \times 1 \times k$ box. One natural type of unfolding of an k -tube is a *river Nile* unfolding, consisting of a $1 \times (2k + 1)$ spine together with $2k$ squares attached symmetrically on either side of the spine. The spine can fold into the end squares and two sides of the k -tube, while appropriately attached squares can form the two remaining sides; but such a river Nile unfolding might fold into other k -cubes as well.

The following result has been verified by exhaustive enumeration.

Proposition 1. *For $0 < k \leq 6$, there exists a common river Nile unfolding of all tree-like incision-free planar k -cubes. For $7 \leq k \leq 8$, there is no common river Nile unfolding of all tree-like incision-free planar k -cubes.*

Indeed, there are 26 common river Nile unfoldings of all tree-like incision-free hexacubes. Figure 7 shows one of them. The question remains open for $k \geq 9$.

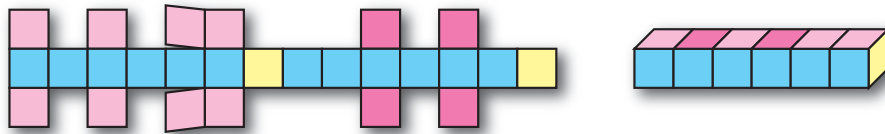


Fig. 7. A river Nile unfolding of all tree-like incision-free hexacubes, and how it folds into a 6-tube.

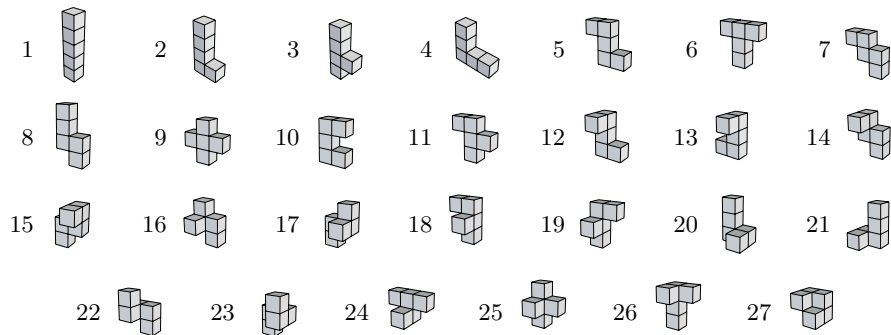
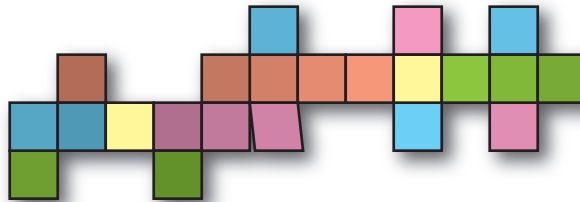


Fig. 8. The 27 incision-free pentacubes. Only those from 1 to 11 are planar. Those from 12 to 23 are chiral twins and are grouped by pairs.

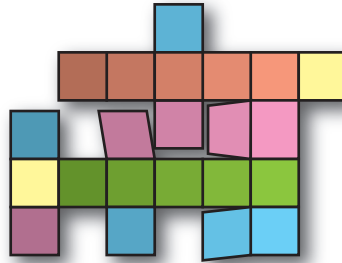
3.3 Pentacubes

The Cubigami puzzle is a common unfolding of all incision-free tetracubes. We investigate the problem of finding common unfoldings of all 27 incision-free pentacubes (shown in Figure 8). The results described in this section are experimental. Using a supercomputer at the Université Libre de Bruxelles, the 1,099,511,627,776 possible unfoldings of the 5-tube (Figure 8(1)) were generated. For each unfolding it was determined which of the remaining pentacubes it could be folded to. This yielded the following discoveries for tree-like incision-free pentacubes:

- There is no common two-sided unfolding of all pentacubes (even when chiral twins are considered identical).
- There are several two-sided common unfoldings for a set of 23 pentacubes (one is pictured below). There is no set of 24 pentacubes that includes the 5-tube and has a common unfolding.
- There are 27 one-sided common unfoldings for a set of 22 pentacubes. There is no set of 23 pentacubes that includes the 5-tube and has a common unfolding.
- There is a unique two-sided unfolding of all 22 non-planar pentacubes (Figure 8(12–27)), shown below.



- There are 492 common one-sided unfoldings of all planar pentacubes (Figure 8(1–11)). None of them can fold to a non-planar pentacube. One such unfolding is shown below.



- There is no common two-sided unfolding of all path-like pentacubes.
- The smallest subset of pentacubes that have no common unfolding has a size of at most 4. For example pentacubes (numbered in Figure 8) 1, 9, 27 taken with any of 6, 7, 12, 13, 24 or 26 have no common unfolding.

It remains open if there is an ℓ such that for all $k \geq \ell$, there are always two incision-free k -ominoes without a common unfolding. We conjecture that this is true. One natural candidate pair to consider was the $(6k+1)$ -tube and k -jack. However, as shown in Figure 9, a common unfolding of the two exists. We still suspect that a modification of the jack, or a series of jacks, might resolve this question.

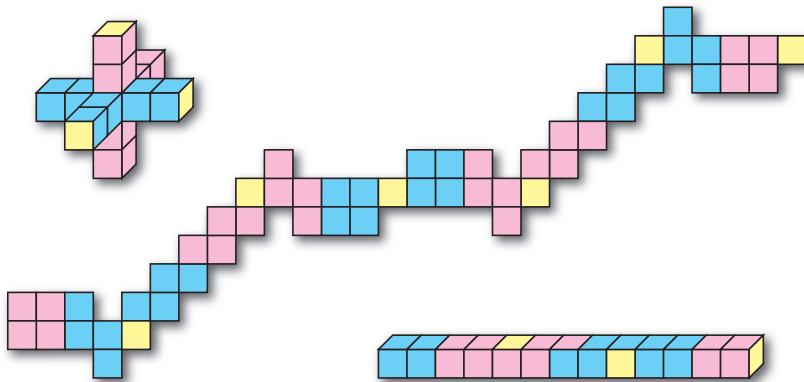


Fig. 9. A common unfolding of the 2-jack and the 13-tube. This generalizes to the k -jack and the $(6k+1)$ -tube.

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